Assignments of Complex Analysis (H)(EN)

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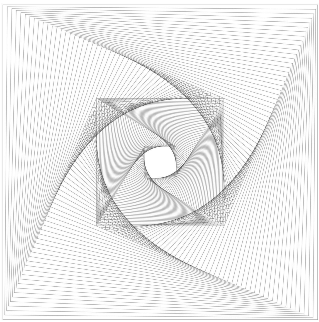


My left-hand-side brain has nothing right; my right-hand-side brain has nothing left.

Contents

Chapter 1 Homework 1

Assignment 1.1 (Exercise 1.4.9)

Show that in polar coordinates, the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta, \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof Let $x = r\cos\theta$ and $y = r\sin\theta$, and write $f = u + iv$ where $u = u(x,y)$, $v = v(x,y)$. Then

$$u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta,\ u_\theta = u_x x_\theta + u_y y_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta).$$

And similarly we have

$$v_r = v_x \cos\theta + v_y \sin\theta,\ v_\theta = r(-v_x \sin\theta + v_y\cos\theta).$$

By the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, we get

$$u_r = v_y\cos\theta - v_x\sin\theta = \frac{1}{r}v_\theta,\ \frac{1}{r}u_\theta = -u_x\sin\theta + u_y\cos\theta = -(v_y\sin\theta + v_x\cos\theta) = -v_r.$$

That is the form we want. Conversely, the above process can be inverted to get $u_x = v_y$ and $u_y = -v_x$, so the two formulations are equivalent when $r > 0$.

Now define $\log z = \log r + i\theta$ in the region $r > 0$ and $-\pi < \theta < \pi$. Then we have $u(r,\theta) = \log r$ and $v(r,\theta) = \theta$. Then

$$u_r = \frac{1}{r},\ u_\theta = 0,\ v_r = 0,\ v_\theta = 1.$$

Therefore we have

$$u_r = \frac{1}{r}v_\theta \text{ and } \frac{1}{r}u_\theta = -v_r.$$

So the Cauchy-Riemann equations hold. Hence $\log z$ is holomorphic in this region.

Assignment 1.2 (Exercise 1.4.10)

Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

*where Δ is the **Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof Firstly, we know that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then we have (for any C^2 function)

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y} - i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2} = \Delta.$$

So we are done.

Assignment 1.3 (Exercise 1.4.13)

Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:
(a) $\mathrm{Re}(f)$ *is constant;*
(b) $\mathrm{Im}(f)$ *is constant;*
(c) $|f|$ *is constant;*
one can conclude that f is constant.

Proof (Assume that Ω is connected.) Write $f = u + iv$, since f is holomorphic in Ω, then by the Cauchy-Riemann equations, we have
(a) If $\mathrm{Re}(f) = u$ is constant, then $u_x = u_y = 0$ in Ω. So, $v_y = u_x = 0$ and $v_x = -u_y = 0$, hence v is constant in Ω. Therefore $f = u + iv$ is constant in Ω.
(b) If $\mathrm{Im}(f) = v$ is constant, then $v_x = v_y = 0$ in Ω. So, $u_y = -v_x = 0$ and $u_x = v_y = 0$, hence u is constant in Ω. Therefore $f = u + iv$ is constant in Ω.
(c) If $|f|$ is constant, let $|f| = c$. If $c = 0$, then $u = v = 0$, so $f \equiv 0$. Now assume $c > 0$, then we have $u^2 + v^2 = c^2$, so $uu_x + vv_x = 0, uu_y + vv_y = 0$. Since $v_x = -u_y$ and $v_y = u_x$, we have $uu_x - vu_y = 0, vu_x + uu_y = 0$. That is

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is $u^2 + v^2 = c^2 > 0$. So $u_x = u_y = 0$. Then, $v_y = u_x = 0, v_x = -u_y = 0$. So both u and v are constant in Ω. Therefore $f = u + iv$ is constant in Ω.

Assignment 1.4 (Exercise 1.4.17)

Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

Proof $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$L - \varepsilon \leqslant \left|\frac{a_{n+1}}{a_n}\right| \leqslant L + \varepsilon.$$

For any $k > N$. Multiplying the inequalities for $n = N, N+1, \cdots, k-1$, then we can obtain

$$(L-\varepsilon)^{k-N} \leqslant \left|\frac{a_k}{a_N}\right| \leqslant (L+\varepsilon)^{k-N}.$$

Thus we have

$$|a_N|(L-\varepsilon)^{k-N} \leqslant |a_k| \leqslant |a_N|(L+\varepsilon)^{k-N}.$$

$$|a_N|^{1/k}(L-\varepsilon)^{\frac{k-N}{k}} \leqslant |a_n|^{1/k} \leqslant |a_N|^{1/k}(L+\varepsilon)^{\frac{k-N}{k}}.$$

Let $k \to \infty$, we have $|a_N|^{1/k} \to 1$ and $\frac{k-N}{k} \to 1$, so

$$L - \varepsilon \leqslant \liminf_{k\to\infty} |a_k|^{1/k} \leqslant \limsup_{k\to\infty} |a_k|^{1/k} \leqslant L + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, let $\varepsilon \to 0$, we can obtain

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

When $L = 0$, it suffices to replace the left-hand side of the initial inequality by 0.

Assignment 1.5 (Exercise 1.4.18)

Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

[Hint: Write $z = z_0 + (z - z_0)$ and use the binomial expansion for z^n.]

Proof We can denote that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

And it has radius of convergence $R > 0$ (It's trivial when $R = 0$). Fix z_0 with $|z_0| < R$. For z near z_0, write $z = z_0 + w$ where $w = z - z_0$. Then by the binomial theorem, we have

$$z^n = (z_0 + w)^n = \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} w^k.$$

Hence, formally, we have

$$f(z_0 + w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} w^k = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) w^k.$$

Define that

$$b_k := \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}.$$

For each fixed $k \geqslant 0$, we first show that b_k is well-defined. Choose r such that $|z_0| < r < R$. Since R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$, we have

$$\sum_{n=0}^{\infty} |a_n| r^n < \infty.$$

Let $\rho := |z_0|/r \in (0, 1)$. Then for $n \geqslant k$, we have

$$\binom{n}{k} |z_0|^{n-k} = r^n \frac{1}{|z_0|^k} \binom{n}{k} \rho^n.$$

Since k is fixed and $0 < \rho < 1$, $\binom{n}{k} \rho^n$ is bounded. Hence there exists a constant $C_k > 0$ such that

$$\binom{n}{k} |z_0|^{n-k} \leqslant C_k r^n, n \geqslant k.$$

Therefore we have

$$\sum_{n=k}^{\infty} \left| a_n \binom{n}{k} z_0^{n-k} \right| \leqslant C_k \sum_{n=k}^{\infty} |a_n| r^n < \infty.$$

Thus b_k is well-defined. Now we claim that the series $\sum_{k=0}^{\infty} b_k w^k$ converges for $|w| < R - |z_0|$ and equals $f(z_0+w)$. Choose r' with $|z_0|+|w|<r'<R$. So for each n, we have

$$\sum_{k=0}^{n}\left|a_n\binom{n}{k}z_0^{n-k}w^k\right| = |a_n|(|z_0|+|w|)^n < |a_n|(r')^n.$$

Because $\sum_{n=0}^{\infty}|a_n|(r')^n<\infty$, so the rearrangement above is valid and we can obtain

$$f(z_0+w)=\sum_{k=0}^{\infty}b_kw^k \text{ for } |w|<r'-|z_0|.$$

Since r' can be chosen arbitrarily with $|z_0|+|w|<r'<R$, the identity holds for all $|w|<R-|z_0|$, and z_0 is arbitrary, so f has a power series expansion around any point in its disc of convergence.

Assignment 1.6 (Exercise 1.4.19)

Prove the following:
(a) The power series $\sum nz^n$ does not converge on any point of the unit circle.
(b) The power series $\sum z^n/n^2$ converges at every point of the unit circle.
(c) The power series $\sum z^n/n$ converges at every point of the unit circle except $z=1$. [Hint: Sum by parts (Abel transformation)]

Proof (a) If $|z|=1$, then the modulus of n th term of this power series is $|nz^n|=n\not\to 0$. Hence this power series dose not converge on any point of the unit circle.
(b) If $|z|=1$, then we have

$$\sum_{n=1}^{\infty}\left|\frac{z^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty.$$

So this power series converges absolutely, hence converges, at every point of the unit circle.
(c) If $z=1$, this power series is $\sum 1/n$, which diverges. Now assume that $|z|=1$ and $z\neq 1$. Let

$$S_N=\sum_{n=1}^{N}z^n=z\frac{1-z^N}{1-z},\ |S_N|\leqslant\frac{2}{|1-z|}.$$

So $\{S_N\}$ is bounded. The sequence $b_n:=1/n$ is monotone decreasing to 0. Hence by Dirichlet's test, this power series converges in $S^1\setminus\{1\}$. On the other hand, by Abel transformation, we have

$$\sum_{n=1}^{N}\frac{z^n}{n}=\frac{S_N}{N}+\sum_{n=1}^{N-1}S_n\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Since $\{S_N\}$ is bounded and $\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=1$, we can conclude convergence in $S^1\setminus\{1\}$.

Assignment 1.7 (Exercise 1.4.20)

Expand $(1-z)^{-m}$ in powers of z. Here m is a fixed positive integer. Also, show that if

$$(1-z)^{-m}=\sum_{n=0}^{\infty}a_nz^n,$$

then one obtains the following asymptotic relation for the coefficients:

$$a_n\sim\frac{1}{(m-1)!}n^{m-1} \text{ as } n\to\infty.$$

Proof Firstly, we know that

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n, |z|<1.$$

Differentiating it $(m-1)$ times, we have

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left(\frac{1}{1-z}\right)=\frac{(m-1)!}{(1-z)^m}, \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left(\sum_{n=0}^{\infty} z^n\right)=\sum_{n=m-1}^{\infty} n(n-1)\cdots(n-m+2)z^{n-m+1}.$$

Let $k=n-m+1\geqslant 0$, for $|z|<1$, we can obtain

$$\frac{1}{(1-z)^m}=\sum_{k=0}^{\infty}\frac{(k+m-1)(k+m-2)\cdots(k+1)}{(m-1)!}z^k=\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}z^k.$$

Thus we have

$$a_n=\binom{n+m-1}{m-1}=\frac{(n+m-1)(n+m-2)\cdots(n+1)}{(m-1)!}=\frac{n^{m-1}}{(m-1)!}\prod_{j=1}^{m-1}\left(1+\frac{j}{n}\right).$$

As $n\to\infty, (1+j/n)\to 1, j=1\cdots,m-1$. Therefore

$$a_n\sim\frac{1}{(m-1)!}n^{m-1} \text{ as } n\to\infty.$$

Assignment 1.8 (Exercise 1.4.23)

Consider the function f defined on $\mathbb{R}$ by

$$f(x)=\begin{cases}0 & \text{if } x\leqslant 0,\\ \mathrm{e}^{-1/x^2} & \text{if } x>0.\end{cases}$$

Prove that f is indefinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0)=0$ for all $n\geqslant 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty}a_nx^n$ for x near the origin. ♠

Proof For $x<0$, $f(x)=0$, hence f is C^∞ there and all derivatives are identically 0. For $x>0$, let $g(x)=\mathrm{e}^{-1/x^2}$. We claim that for each $n\geqslant 0$ there exists a polynomial P_n such that

$$g^{(n)}(x)=P_n\left(\frac{1}{x}\right)\mathrm{e}^{-1/x^2}, x>0.$$

This holds when $n=0$ and $P_0\equiv 1$. If it holds for n, then differentiate it, we have

$$g^{(n+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n\left(\frac{1}{x}\right)\mathrm{e}^{-1/x^2}\right)=\left(-\frac{1}{x^2}P_n'\left(\frac{1}{x}\right)+\frac{2}{x^3}P_n\left(\frac{1}{x}\right)\right)\mathrm{e}^{-1/x^2}:=P_{n+1}\left(\frac{1}{x}\right)\mathrm{e}^{-1/x^2}.$$

So the claim is proved. Thus g is C^∞ on $(0,\infty)$. Since P_n are all polynomials, we can obtain

$$\lim_{x\to 0^+} g^{(n)}(x)=0, n\geqslant 0.$$

Therefore $f\in C^\infty(\mathbb{R})$ with $f^{(n)}(0)=0$ for all $n\geqslant 1$. Finally, if f has a converging power series expansion near the origin, let

$$f(x)=\sum_{n=0}^{\infty}a_nx^n, |x|<R,$$

then we would have $a_n=(f^{(n)}(0))/n!=0$ for all $n\geqslant 1$, and also $a_0=f(0)=0$. Hence the series would be identically 0, implying $f(x)\equiv 0$ when $|x|<R$, which contradicts $f(x)=e^{-1/x^2}>0$ for $x\in(0,R)$. Therefore f does not have a converging power series expansion near the origin.

Chapter 2 Homework 2

Assignment 2.1 (Exercise 1.4.21)

Show that for $|z| < 1$, one has

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z},$$

and

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}.$$

Justify any change in the order of summation.

[Hint: Use the dyadic expansion of an integer and the fact that $2^{k+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^k$.]

Proof For the first identity, since $|z| < 1$, for every $n \geqslant 0$ we have

$$\frac{z^{2^n}}{1-z^{2^{n+1}}} = \sum_{m=0}^{\infty} z^{2^n + m2^{n+1}}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{1-z^{2^{n+1}}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^{(2m+1)2^n}.$$

Now every positive integer N can be written uniquely as $N = 2^n(2m+1)$ with $n, m \geqslant 0$. Therefore each monomial z^N appears exactly once in the above double sum, so formally the sum equals

$$\sum_{N=1}^{\infty} z^N = \frac{z}{1-z}.$$

It remains to justify the rearrangement. We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left|z^{(2m+1)2^n}\right| \leqslant \sum_{N=1}^{\infty} |z|^N < \infty.$$

This is because $(2m+1)2^n$ are all distinct positive integers. So the double series converges absolutely, and the change in the order of summation is justified.

For the second identity, for $|z| < 1$, we have

$$\frac{2^k z^{2^k}}{1+z^{2^k}} = 2^k z^{2^k} \sum_{m=0}^{\infty} (-1)^m z^{m2^k} = \sum_{m=1}^{\infty} 2^k (-1)^{m-1} z^{m2^k}.$$

Therefore we obtain

$$\sum_{k=0}^{\infty} \frac{2^k z^{2^k}}{1+z^{2^k}} = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} 2^k (-1)^{m-1} z^{m2^k}.$$

Fix $N \geqslant 1$, and write $N = 2^k m, 0 \leqslant k \leqslant v_2(N) := r$ and let $q = N/2^r$. So the coefficient of z^N is

$$\sum_{k:2^k|N} 2^k (-1)^{N/2^k - 1} = \sum_{k=0}^{r} 2^k (-1)^{2^{r-k}q-1}.$$

If $k < r$, then $2^{r-k}q$ is even, so $(-1)^{2^{r-k}q-1} = -1$; if $k = r$, then $2^{r-r}q = q$ is odd, so $(-1)^{q-1} = 1$.

Hence the coefficient is

$$-\sum_{k=0}^{r-1} 2^k + 2^r = -(2^r - 1) + 2^r = 1.$$

Therefore every z^N has coefficient 1, $N \geqslant 1$, and so

$$\sum_{k=0}^{\infty} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \sum_{N=1}^{\infty} z^N = \frac{z}{1-z}.$$

Again, we show absolute convergence to justify the rearrangement, since

$$\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \left| 2^k (-1)^{m-1} z^{m2^k} \right| = \sum_{k=0}^{\infty} 2^k \sum_{m=1}^{\infty} |z|^{m2^k} = \sum_{k=0}^{\infty} \frac{2^k |z|^{2^k}}{1 - |z|^{2^k}}.$$

For all sufficiently large k we have $|z|^{2^k} \leqslant 1/2$, so

$$\frac{2^k |z|^{2^k}}{1 - |z|^{2^k}} \leqslant 2^{k+1} |z|^{2^k}.$$

Since we have

$$2 \sum_{k=0}^{\infty} 2^k |z|^{2^k} \leqslant 2 \sum_{n=1}^{\infty} n|z|^n = \frac{2|z|}{(1-|z|)^2} < \infty.$$

So the double series converges absolutely. So we are done.

Assignment 2.2 (Exercise 1.4.24)

Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a, b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reverse orientation. Prove that for any continuous function f on γ

$$\int_{\gamma} f(z) \mathrm{d}z = -\int_{\gamma^-} f(z) \mathrm{d}z.$$

Proof Let $\widetilde{z}(t) = z(a + b - t)$, $t \in [a, b]$. Then $\widetilde{z}$ is a parametrization of γ^-. By definition we have

$$\int_{\gamma^-} f(z) \mathrm{d}z = \int_a^b f(\widetilde{z}(t)) \widetilde{z}'(t) \mathrm{d}t.$$

Since $\widetilde{z}'(t) = -z'(a + b - t)$, we can obtain

$$\int_{\gamma^-} f(z) \mathrm{d}z = -\int_a^b f(z(a + b - t)) z'(a + b - t) \mathrm{d}t.$$

Now let $u = a + b - t$, then $\mathrm{d}u = -\mathrm{d}t$, and when $t = a$, $u = b$, while when $t = b$, $u = a$. Thus

$$\int_{\gamma^-} f(z) \mathrm{d}z = -\int_b^a f(z(u)) z'(u)(-\mathrm{d}u) = -\int_a^b f(z(u)) z'(u) \mathrm{d}u = -\int_{\gamma} f(z) \mathrm{d}z.$$

This holds for any continuous function f on γ. So we are done.

Assignment 2.3 (Exercise 1.4.26)

Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

Proof Let F and G be two primitives of f in Ω. Then we have $F'(z) = G'(z) = f(z), \forall z \in \Omega$. Hence $(F - G)'(z) = 0, \forall z \in \Omega$. Let $H = F - G$. Then H is holomorphic and $H' \equiv 0$ in Ω. Hence by Corollary 3.4 of the textbook we have H is constant, i.e., $F - G$ is constant, so any two primitives of f differ by a constant. So we are done.

Assignment 2.4 (Exercise 2.6.1)

Prove that

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \int_0^\infty \cos(x^2)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

These are the ***Fresnel integrals****. Here,* $\int_0^\infty$ *is interpreted as* $\lim_{R\to\infty}\int_0^R$.

[Hint: Integrate the function e^{-z^2} *over the path in Figure 1. Recall that* $\int_{-\infty}^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$*.]*

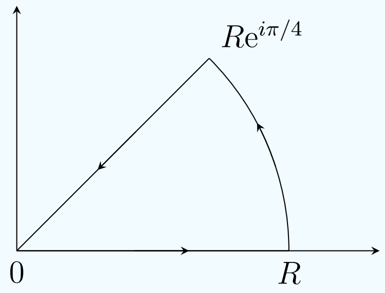


Figure 1. *The contour in Exercise 1*

Proof For $R > 0$, let Γ_R be the boundary of $\{r\mathrm{e}^{i\theta} : 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant \pi/4\}$, consisting of the three curves $\gamma_1(t) = t, 0 \leqslant t \leqslant R$; $\gamma_2(\theta) = R\mathrm{e}^{i\theta}, 0 \leqslant \theta \leqslant \pi/4$; $\gamma_3(t) = t\mathrm{e}^{i\pi/4}, R \geqslant t \geqslant 0$. Since e^{-z^2} is entire, Cauchy's theorem gives

$$\int_{\Gamma_R} \mathrm{e}^{-z^2}\mathrm{d}z = 0.$$

Therefore

$$\int_{\gamma_1} \mathrm{e}^{-z^2}\mathrm{d}z + \int_{\gamma_2} \mathrm{e}^{-z^2}\mathrm{d}z + \int_{\gamma_3^-} \mathrm{e}^{-z^2}\mathrm{d}z = 0.$$

Firstly, we have

$$\int_{\gamma_1} \mathrm{e}^{-z^2}\mathrm{d}z = \int_0^R \mathrm{e}^{-x^2}\mathrm{d}x.$$

Also, along γ_3, writing $z = t\mathrm{e}^{i\pi/4}$, we have $z^2 = t^2\mathrm{e}^{i\pi/2} = it^2, \mathrm{d}z = \mathrm{e}^{i\pi/4}\mathrm{d}t$, so

$$\int_{\gamma_3^-} \mathrm{e}^{-z^2}\mathrm{d}z = -\int_0^R \mathrm{e}^{-it^2}\mathrm{e}^{i\pi/4}\mathrm{d}t.$$

Hence

$$\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x + \int_{\gamma_2} \mathrm{e}^{-z^2}\mathrm{d}z = \mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-it^2}\mathrm{d}t.$$

We now show that

$$\lim_{R\to\infty}\int_{\gamma_2} \mathrm{e}^{-z^2}\mathrm{d}z = 0.$$

Indeed, on γ_2, we can write $z = R\mathrm{e}^{i\theta}, 0 \leqslant \theta \leqslant \pi/4$ so that

$$\mathrm{d}z = iR\mathrm{e}^{i\theta}\mathrm{d}\theta, |\mathrm{e}^{-z^2}| = \mathrm{e}^{-R^2\cos(2\theta)}.$$

Hence we can obtain

$$\left|\int_{\gamma_2} \mathrm{e}^{-z^2}\mathrm{d}z\right| \leqslant R\int_0^{\pi/4} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta.$$

Now let $\phi = \pi/2 - 2\theta$. Then $\cos(2\theta) = \sin\phi$ and $\mathrm{d}\theta = -\frac{1}{2}\mathrm{d}\phi$, so

$$R\int_0^{\pi/4} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta = \frac{R}{2}\int_0^{\pi/2} \mathrm{e}^{-R^2\sin\phi}\mathrm{d}\phi.$$

Since we have

$$\sin\phi \geqslant \frac{2}{\pi}\phi, 0 \leqslant \phi \leqslant \frac{\pi}{2},$$

we can obtain

$$\frac{R}{2}\int_0^{\pi/2} \mathrm{e}^{-R^2\sin\phi}\mathrm{d}\phi \leqslant \frac{R}{2}\int_0^{\pi/2} \mathrm{e}^{-(2/\pi)R^2\phi}\mathrm{d}\phi \leqslant \frac{R}{2}\int_0^{\infty} \mathrm{e}^{-(2/\pi)R^2\phi}\mathrm{d}\phi = \frac{\pi}{4R} \to 0\,(R\to\infty).$$

Therefore we have

$$\lim_{R\to\infty}\int_{\gamma_2} \mathrm{e}^{-z^2}\mathrm{d}z = 0.$$

Let $R \to \infty$, and use

$$\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

We can obtain

$$\mathrm{e}^{i\pi/4}\int_0^\infty \mathrm{e}^{-it^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Hence we have

$$\int_0^\infty \mathrm{e}^{-it^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-i\pi/4} = \frac{\sqrt{\pi}}{2}\cdot\frac{1-i}{\sqrt{2}} = \frac{\sqrt{2\pi}}{4}(1-i).$$

Taking real and imaginary parts gives

$$\int_0^\infty \cos(x^2)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}, -\int_0^\infty \sin(x^2)\mathrm{d}x = -\frac{\sqrt{2\pi}}{4}.$$

Therefore we obtain

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \int_0^\infty \cos(x^2)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

So we are done.

Assignment 2.5 (Exercise 2.6.2)

Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$.

[Hint: The integral equals $\displaystyle\frac{1}{2i}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ix}-1}{x}\mathrm{d}x$. *Use the indented semicircle.]*

Proof Denote that

$$I := \frac{1}{2i}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ix}-1}{x}\mathrm{d}x.$$

Since we have

$$\frac{\mathrm{e}^{ix}-1}{x} = \frac{\cos x - 1}{x} + i\frac{\sin x}{x}.$$

And $(\cos x - 1)/x$ is odd, we can obtain

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{d}x = \int_0^{\infty} \frac{\sin x}{x}\mathrm{d}x.$$

Thus it suffices to prove that $I = \pi/2$. Fix $R > \varepsilon > 0$, and consider the contour consisting of the interval $[-R, -\varepsilon]$, the small upper semicircle $C_\varepsilon : z = \varepsilon\mathrm{e}^{i\theta}, \pi \geqslant \theta \geqslant 0$, the interval $[\varepsilon, R]$, and the large upper semicircle $C_R : z = R\mathrm{e}^{i\theta}, 0 \leqslant \theta \leqslant \pi$. Since we have

$$f(z) = \frac{\mathrm{e}^{iz} - 1}{z}$$

is entire because the singularity at $z = 0$ is removable. Hence the integral over the contour is 0, i.e.,

$$\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix} - 1}{x}\mathrm{d}x + \int_{C_\varepsilon^-} \frac{\mathrm{e}^{iz} - 1}{z}\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{ix} - 1}{x}\mathrm{d}x + \int_{C_R} \frac{\mathrm{e}^{iz} - 1}{z}\mathrm{d}z = 0.$$

Since we have

$$\int_{C_R} \frac{1}{z}\mathrm{d}z = i\pi, \int_{C_\varepsilon^-} \frac{1}{z}\mathrm{d}z = -i\pi, \int_{-R}^{-\varepsilon} \frac{1}{x}\mathrm{d}x + \int_{\varepsilon}^{R} \frac{1}{x}\mathrm{d}x = 0.$$

we can obtain

$$\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix}}{x}\mathrm{d}x + \int_{C_\varepsilon^-} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{ix}}{x}\mathrm{d}x + \int_{C_R} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z = 0.$$

Firstly, on C_R, we have $z = R\mathrm{e}^{i\theta}, 0 \leqslant \theta \leqslant \pi$, then $\mathrm{d}z = iR\mathrm{e}^{i\theta}\mathrm{d}\theta$ and hence

$$\left|\int_{C_R} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z\right| = \left|i\int_0^{\pi} \mathrm{e}^{iR\mathrm{e}^{i\theta}}\mathrm{d}\theta\right| \leqslant \int_0^{\pi} \left|\mathrm{e}^{iR\mathrm{e}^{i\theta}}\right| \mathrm{d}\theta = \int_0^{\pi} \mathrm{e}^{-R\sin\theta}\mathrm{d}\theta = 2\int_0^{\pi/2} \mathrm{e}^{-R\sin\theta}\mathrm{d}\theta.$$

Since for $0 \leqslant \theta \leqslant \pi/2$, we have $\sin\theta \geqslant 2\theta/\pi$, hence

$$2\int_0^{\pi/2} \mathrm{e}^{-R\sin\theta}\mathrm{d}\theta \leqslant 2\int_0^{\pi/2} \mathrm{e}^{-2R\theta/\pi}\mathrm{d}\theta = \frac{\pi}{R}(1 - \mathrm{e}^{-R}) \to 0\,(R \to \infty) \Rightarrow \lim_{R\to\infty}\int_{C_R} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z = 0.$$

Next, on C_ε, we have $z = \varepsilon\mathrm{e}^{i\theta}, \pi \geqslant \theta \geqslant 0$ and hence

$$\int_{C_\varepsilon^-} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z = \int_{\pi}^{0} i\mathrm{e}^{i\varepsilon\mathrm{e}^{i\theta}}\mathrm{d}\theta.$$

Since we have $|\mathrm{e}^{i\varepsilon\mathrm{e}^{i\theta}}| = \mathrm{e}^{-\varepsilon\sin\theta} \in [\mathrm{e}^{-\varepsilon}, \mathrm{e}^{\varepsilon}]$ and $\arg(\mathrm{e}^{i\varepsilon\mathrm{e}^{i\theta}}) = \varepsilon\cos\theta \in [-\varepsilon, \varepsilon]$. Hence we can obtain $\mathrm{e}^{i\varepsilon\mathrm{e}^{i\theta}} \to 1$ uniformly in θ, so we have

$$\int_{C_\varepsilon^-} \frac{\mathrm{e}^{iz}}{z}\mathrm{d}z \to \int_{\pi}^{0} i\mathrm{d}\theta = -i\pi.$$

Letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0^+}} \left(\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix}}{x}\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{ix}}{x}\mathrm{d}x\right) = i\pi.$$

Since we have

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0^+}} \left(\int_{-R}^{-\varepsilon} \frac{1}{x}\mathrm{d}x + \int_{\varepsilon}^{R} \frac{1}{x}\mathrm{d}x\right) = 0.$$

Then we can obtain

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0^+}} \left(\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix} - 1}{x}\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{ix} - 1}{x}\mathrm{d}x\right) = i\pi.$$

That is $2iI = i\pi$, $I = \pi/2$. So we are done.

Assignment 2.6 (Exercise 2.6.3)

Evaluate the integrals

$$\int_0^\infty \mathrm{e}^{-ax}\cos bx\mathrm{d}x \text{ and } \int_0^\infty \mathrm{e}^{-ax}\sin bx\mathrm{d}x, a>0$$

by integrating e^{-Az}, $A=\sqrt{a^2+b^2}$, *over an appropriate sector with angle* ω, *with* $\cos\omega = a/A$. ♠

Solution *Let* $A=\sqrt{a^2+b^2}, \cos\omega = a/A, \sin\omega = b/A$. *Assume that* $\omega\in(0,\pi/2)$ *WLOG. Then* $a = A\cos\omega, b = A\sin\omega$. *For* $R>0$, *consider the sector contour* Γ_R *consisting of* $[0,R], \{R\mathrm{e}^{i\theta}: 0\leqslant\theta\leqslant\omega\}, \{t\mathrm{e}^{i\omega}: R\geqslant t\geqslant 0\}$. *Since* e^{-Az} *is entire, we have*

$$\int_{\Gamma_R}\mathrm{e}^{-Az}\mathrm{d}z = 0.$$

Thus

$$\int_0^R \mathrm{e}^{-Ax}\mathrm{d}x + \int_{C_R}\mathrm{e}^{-Az}\mathrm{d}z - \int_0^R \mathrm{e}^{-Ate^{i\omega}}\mathrm{e}^{i\omega}\mathrm{d}t = 0.$$

Now we claim that

$$\lim_{R\to\infty}\int_{C_R}\mathrm{e}^{-Az}\mathrm{d}z = 0.$$

Indeed, on C_R, $z = R\mathrm{e}^{i\theta}$ *with* $0\leqslant\theta\leqslant\omega$, *so* $|\mathrm{e}^{-Az}| = \mathrm{e}^{-AR\cos\theta}\leqslant \mathrm{e}^{-AR\cos\omega} = \mathrm{e}^{-aR}$, *then*

$$\left|\int_{C_R}\mathrm{e}^{-Az}\mathrm{d}z\right| \leqslant R\omega\mathrm{e}^{-aR}\to 0\,(R\to\infty) \Rightarrow \lim_{R\to\infty}\int_{C_R}\mathrm{e}^{-Az}\mathrm{d}z = 0.$$

The claim is proved. Then letting $R\to\infty$, *we obtain*

$$\int_0^\infty \mathrm{e}^{-Ax}\mathrm{d}x = \mathrm{e}^{i\omega}\int_0^\infty \mathrm{e}^{-Ate^{i\omega}}\mathrm{d}t.$$

Since $A\mathrm{e}^{i\omega} = A(\cos\omega + i\sin\omega) = a+ib$, *so* $\mathrm{e}^{-Ate^{i\omega}} = \mathrm{e}^{-(a+ib)t} = \mathrm{e}^{-at}(\cos bt - i\sin bt)$. *Also*

$$\int_0^\infty \mathrm{e}^{-Ax}\mathrm{d}x = \frac{1}{A}.$$

Therefore

$$\int_0^\infty \mathrm{e}^{-at}(\cos bt - i\sin bt)\mathrm{d}t = \frac{\mathrm{e}^{-i\omega}}{A} = \frac{\cos\omega - i\sin\omega}{A} = \frac{a-ib}{A^2} = \frac{a-ib}{a^2+b^2}.$$

Taking real and imaginary parts, we obtain

$$\int_0^\infty \mathrm{e}^{-ax}\cos bx\mathrm{d}x = \frac{a}{a^2+b^2}, \int_0^\infty \mathrm{e}^{-ax}\sin bx\mathrm{d}x = \frac{b}{a^2+b^2}.$$

Assignment 2.7 (Exercise 2.6.4)

Prove that for all $\xi\in\mathbb{C}$, *we have*

$$\mathrm{e}^{-\pi\xi^2} = \int_{-\infty}^\infty \mathrm{e}^{-\pi x^2}\mathrm{e}^{2\pi ix\xi}\mathrm{d}x.$$

♠

Proof Fix $\xi\in\mathbb{C}$, and denote that $RHS := I(\xi)$. Since we have $-\pi x^2 + 2\pi ix\xi = -\pi(x^2 - 2ix\xi) = -\pi(x-i\xi)^2 - \pi\xi^2$, hence we can obtain

$$I(\xi) = \mathrm{e}^{-\pi\xi^2}\int_{-\infty}^\infty \mathrm{e}^{-\pi(x-i\xi)^2}\mathrm{d}x.$$

If we claim that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x-i\xi)^2}\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{d}x.$$

Then we can obtain the original equality by

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\cdot\sqrt{\pi} = 1.$$

Now we prove the claim. Let $f(z) = \mathrm{e}^{-\pi z^2}$, which is entire. For $R > 0$, assume that $-i\xi$ is in the upper half of complex plane WLOG. Consider the parallelogram with vertices $-R, R, R-i\xi, -R-i\xi$, Connect these four points in a counterclockwise order with straight lines to obtain Γ_R. Then by Cauchy's theorem, we have

$$\int_{\Gamma_R} f(z)\mathrm{d}z = 0.$$

Writing this integral as the sum over the four sides, we obtain

$$\int_{-R}^{R} \mathrm{e}^{-\pi x^2}\mathrm{d}x + \int_0^1 \mathrm{e}^{-\pi(R-it\xi)^2}(-i\xi)\mathrm{d}t - \int_{-R}^{R} \mathrm{e}^{-\pi(x-i\xi)^2}\mathrm{d}x + \int_0^1 \mathrm{e}^{-\pi(-R-i\xi+it\xi)^2}(i\xi)\mathrm{d}t = 0.$$

Write $\xi = a+ib$, then $R-it\xi = (R+tb)-ita$, so $\mathrm{Re}((R-it\xi)^2) = (R+tb)^2-t^2a^2$. Hence we have $|\mathrm{e}^{-\pi(R-it\xi)^2}| = \mathrm{e}^{-\pi\,\mathrm{Re}((R-it\xi)^2)} = \mathrm{e}^{-\pi((R+tb)^2-t^2a^2)}$. Since $t\in[0,1]$, for sufficiently large R we have $(R+tb)^2-t^2a^2 = R^2+2Rtb+t^2(b^2-a^2) \geqslant R^2-2R|b|-|b^2-a^2| \geqslant R^2-2R|b|-|\xi|^2 \geqslant R^2/2-|\xi|^2$ and hence $|\mathrm{e}^{-\pi(R-it\xi)^2}| \leqslant \mathrm{e}^{\pi|\xi|^2}\mathrm{e}^{-\pi R^2/2}$. Thus

$$\left|\int_0^1 \mathrm{e}^{-\pi(R-it\xi)^2}(-i\xi)\mathrm{d}t\right| \leqslant |\xi|\mathrm{e}^{\pi|\xi|^2}\mathrm{e}^{-\pi R^2/2} \to 0\,(R\to\infty) \Rightarrow \lim_{R\to\infty}\int_0^1 \mathrm{e}^{-\pi(R-it\xi)^2}(-i\xi)\mathrm{d}t = 0.$$

Similarly we have

$$\lim_{R\to\infty}\int_0^1 \mathrm{e}^{-\pi(-R-i\xi+it\xi)^2}(i\xi)\mathrm{d}t = 0.$$

Letting $R\to\infty$, we can obtain

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x-i\xi)^2}\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{d}x.$$

This proves the claim, and therefore $I(\xi) = \mathrm{e}^{-\pi\xi^2}$. So we are done.

Chapter 3 Homework 3

Assignment 3.1 (Exercise 2.6.6)

Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z)\mathrm{d}z = 0.$$

Proof Let C_ε be the circle centered at w with radius ε and contained inside T (anticlockwise). Since f is holomorphic in the region between T and C_ε, Cauchy's theorem gives

$$\int_T f(z)\mathrm{d}z = \int_{C_\varepsilon} f(z)\mathrm{d}z.$$

Since f is bounded near w, so $\exists M > 0$ such that $|f(z)| \leqslant M$ on C_ε for all sufficiently small ε. So

$$\left|\int_{C_\varepsilon} f(z)\mathrm{d}z\right| \leqslant 2\pi\varepsilon M \to 0$$

as $\varepsilon \to 0^+$. Therefore we have $\displaystyle\int_T f(z)\mathrm{d}z = 0$. So we are done.

Assignment 3.2 (Exercise 2.6.7)

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leqslant d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Note. *In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.*

[Hint: $2f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=r} \dfrac{f(\zeta) - f(-\zeta)}{\zeta^2}\mathrm{d}\zeta$ whenever $0 < r < 1$.]

Proof Fix $0 < r < 1$. Let $C_r = \{\zeta \in \mathbb{C} : |\zeta| = r\}$ (anticlockwise), Cauchy's integral formula gives

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta^2}\mathrm{d}\zeta,$$

Replacing ζ by $-\zeta$, we can obtain

$$f'(0) = -\frac{1}{2\pi i}\int_{C_r} \frac{f(-\zeta)}{\zeta^2}\mathrm{d}\zeta.$$

Hence we have

$$2f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\mathrm{d}\zeta.$$

Therefore

$$2|f'(0)| \leqslant \frac{1}{2\pi}\int_{C_r} \frac{|f(\zeta) - f(-\zeta)|}{|\zeta|^2}|\mathrm{d}\zeta| \leqslant \frac{d}{2\pi}\int_{C_r} \frac{|\mathrm{d}\zeta|}{r^2} = \frac{d}{r}.$$

Letting $r \to 1^-$ gives $2|f'(0)| \leqslant d$. It is obvious that equality holds when f is linear. We now prove the converse. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ near the origin and define

$$N(r) := \frac{1}{\pi r^2} \int_{B(0,r)} |f'(z)|^2 \mathrm{d}x\mathrm{d}y.$$

If $f'(0) = 0$, then $d = 0$, so f is constant. Hence we can assume $f'(0) \neq 0$, then $N(0^+) = |f'(0)|^2 > 0$. So f is locally injective near the origin. Then by expanding f' and the area formula, we have

$$N(r) = \frac{\mathrm{Area}(f(B(0,r)))}{\pi r^2} = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n \rho^{n-1} \mathrm{e}^{i(n-1)\theta} \right|^2 \rho \mathrm{d}\theta \mathrm{d}\rho = \sum_{n=1}^{\infty} n |a_n|^2 r^{2n-2},$$

$$N'(r) = \sum_{n=2}^{\infty} n(2n-2)|a_n|^2 r^{2n-3}.$$

If f is not linear, then there exists $a_n \neq 0$ with $n \geqslant 2$, so $N'(r) > 0$ for all sufficiently small r. Hence for sufficiently small r, we have $|f'(0)|^2 = N(0^+) < N(r)$. And the isodiametric inequality gives

$$\mathrm{Area}(f(B(0,r))) \leqslant \pi \left(\frac{d(r)}{2} \right)^2,$$

where $d(r)$ is the diameter of $f(B(0,r))$. Therefore

$$|f'(0)|^2 < N(r) \leqslant \left(\frac{d(r)}{2r} \right)^2.$$

On the other hand, by the maximum modulus principle, we have

$$\frac{d(r)}{r} = \sup_{\theta \in [0,2\pi]} \sup_{|z|=r} \left| \frac{f(\mathrm{e}^{i\theta} z) - f(z)}{z} \right|.$$

For each fixed θ, the function $(f(\mathrm{e}^{i\theta} z) - f(z))/z$ is holomorphic in $\mathbb{D}$ because the singularity at $z = 0$ is removable, so by the maximum modulus principle, its maximum on $|z| = r$ is nondecreasing in r. Taking the supremum over θ, we obtain that $d(r)/r$ is nondecreasing in r. So if the equality holds in $2|f'(0)| \leqslant d$, then for small r we would have

$$\frac{d(r)}{r} \leqslant \frac{d(1)}{1} = d = 2|f'(0)|,$$

which contradicts the strict inequality above. Therefore f must be linear.

Assignment 3.3 (Exercise 2.6.8)

If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leqslant A(1 + |z|)^{\eta}, \eta \text{ a fixed real number},$$

for all z in that strip, show that for each integer $n \geqslant 0$ there exists $A_n \geqslant 0$ so that

$$|f^{(n)}(x)| \leqslant A_n (1 + |x|)^{\eta}, \text{ for all } x \in \mathbb{R}.$$

[Hint: Use the Cauchy inequalities.]

Proof Fix $n \geqslant 0$ and $x \in \mathbb{R}$. Then $\overline{B(x, 1/2)}$ is contained in the strip, so Cauchy inequalities give

$$|f^{(n)}(x)| \leqslant n! 2^n \sup_{|z-x|=1/2} |f(z)|.$$

Now if $|z - x| = 1/2$, then $|z| \leqslant |x| + 1/2$, hence $1 + |z| \leqslant 3(1 + |x|)/2$. Also $|z| \geqslant |x| - 1/2$, so $1 + |z| \geqslant (1 + |x|)/2$. Therefore there exists a constant $C_\eta > 0$, depending only on η, such that

$(1+|z|)^\eta \leqslant C_\eta(1+|x|)^\eta$ on $|z-x| = 1/2$. It follows that $|f(z)| \leqslant AC_\eta(1+|x|)^\eta$ on the circle $\{z \in \mathbb{C} : |z-x| = 1/2\}$, and so $|f^{(n)}(x)| \leqslant n!2^n AC_\eta(1+|x|)^\eta$. Thus the conclusion holds with $A_n = n!2^n AC_\eta$. So we are done.

Assignment 3.4 (Exercise 2.6.9)

Let Ω be a bounded connected open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \text{ and } \varphi'(z_0) = 1$$

then φ is linear.

[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here we use the standard O notation, where $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \leqslant C|g(z)|$ for some constant C as $|z| \to 0$.]

♠

Proof By replacing φ with $z \mapsto \varphi(z+z_0) - z_0$ on $\Omega - z_0$, we can assume that $z_0 = 0$. Suppose, for contradiction, that φ is not linear. Then there exist $n \geqslant 2$ with $a_n \neq 0$ such that $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \cdots \circ \varphi$, where φ appears k times. We claim that $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$ near 0. This is proved by induction on k. The case $k = 1$ is clear. If it holds for k, then $\varphi_{k+1}(z) = \varphi(\varphi_k(z)) = \varphi_k(z) + a_n\varphi_k(z)^n + O(z^{n+1})$. Since $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$, we have $\varphi_k(z)^n = z^n + O(z^{n+1})$, and therefore $\varphi_{k+1}(z) = z + (k+1)a_n z^n + O(z^{n+1})$. So the claim is proved. Now each φ_k maps Ω into Ω, and Ω is bounded, so there exists $M > 0$ such that $|\varphi_k| \leqslant M$ on Ω for all k. Choose $r > 0$ such that $\overline{B(0,r)} \subset \Omega$. By Cauchy inequalities, we have $|\varphi_k^{(n)}(0)| \leqslant n!M/r^n$. But from the expansion of φ_k we have $\varphi_k^{(n)}(0) = n!ka_n$. Hence $|a_n| \leqslant M/(kr^n)$ for every $k \geqslant 1$. Letting $k \to \infty$, we get $a_n = 0$, which leads to a contradiction. Therefore φ must be linear. So we are done.

Assignment 3.5 (Exercise 2.6.10)

Weierstrass's theorem states that a continuous function on $[0,1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

♠

Solution *No. We prove the following lemma firstly.*

Lemma 3.1

Let f be a continuous function on the closed unit disc $\overline{\mathbb{D}}$. If there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $\overline{\mathbb{D}}$. Then f is holomorphic on the unit disc $\mathbb{D}$.

♡

Proof Let $z_0 \in \mathbb{D}$. Choose $r > 0$ such that $\overline{B(z_0,r)} \subset \mathbb{D}$, and let $\Gamma = \{\zeta \in \mathbb{C} : |\zeta - z_0| = r\}$. Since each p_n is a polynomial, it is holomorphic on $\mathbb{C}$. Hence Cauchy's integral formula gives

$$p_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{p_n(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

for every $z \in B(z_0, r)$. Now fix ρ with $0 < \rho < r$. Since $p_n \to f$ uniformly on $\overline{\mathbb{D}}$, in particular $p_n \to f$ uniformly on the circle Γ. Also, for $\zeta \in \Gamma$ and $z \in \overline{B(z_0, \rho)}$, we have $|\zeta - z| \geqslant r - \rho > 0$. Thus we can obtain

$$\frac{p_n(\zeta)}{\zeta - z} \to \frac{f(\zeta)}{\zeta - z}$$

uniformly on $\Gamma \times \overline{B(z_0, \rho)}$. So we can pass the limit through the integral sign and obtain

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

for every $z \in \overline{B(z_0, \rho)}$. Now we claim that f is holomorphic on $B(z_0, \rho)$. For $z \in B(z_0, \rho)$ and for $h \in \mathbb{C}$ with $|h|$ is sufficiently small, we have $z + h \in B(z_0, \rho)$ and $|\zeta - z - h| \geqslant (r - \rho)/2$, then

$$\left| \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \right| = \frac{1}{2\pi} \left| \int_\Gamma \left(\frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2} \right) \mathrm{d}\zeta \right|$$

$$\leqslant \frac{1}{2\pi} \int_\Gamma \frac{\sup_{\zeta \in \Gamma} |f(\zeta)|}{r - \rho} \left| \frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right| |\mathrm{d}\zeta| \leqslant \frac{2r \sup_{\zeta \in \Gamma} |f(\zeta)|}{(r - \rho)^3} |h| \to 0 \, (h \to 0).$$

Therefore we have

$$f'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta$$

for every $z \in B(z_0, \rho)$. Thus f is holomorphic near z_0. Since z_0 is chosen arbitrarily, f is holomorphic on $\mathbb{D}$ and the lemma is proved.

Now let's return to the original question. Consider the function $f(z) = \bar{z}$ on $\overline{\mathbb{D}}$. It is continuous on $\overline{\mathbb{D}}$. If there were polynomials $p_n(z)$ such that $p_n \to f(z)$ uniformly on $\overline{\mathbb{D}}$, then $f(z) = \bar{z}$ would be holomorphic on $\mathbb{D}$ by the lemma, which leads to a contradiction. So the answer is negative.

Assignment 3.6 (Exercise 2.6.11)

Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\varphi}) \operatorname{Re} \left(\frac{R\mathrm{e}^{i\varphi} + z}{R\mathrm{e}^{i\varphi} - z} \right) \mathrm{d}\varphi.$$

(b) Show that

$$\operatorname{Re} \left(\frac{R\mathrm{e}^{i\gamma} + r}{R\mathrm{e}^{i\gamma} - r} \right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

[Hint: For the first part, note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy's integral formula, to deduce the desired identity.]

Proof (a) Let $\zeta = R\mathrm{e}^{i\varphi}$. Then $\mathrm{d}\zeta = i\zeta \mathrm{d}\varphi$, so it is sufficient to prove

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = R} f(\zeta) \operatorname{Re} \left(\frac{\zeta + z}{\zeta - z} \right) \frac{\mathrm{d}\zeta}{\zeta}.$$

Let $w = R^2/\bar{z}$. Since $|z| < R$, we have $|w| > R$, so $\zeta \mapsto f(\zeta)/(\zeta - w)$ is holomorphic on D_R. So

$$\int_{|\zeta| = R} \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta = 0.$$

Combining this with the Cauchy's integral formula, we obtain

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta-z}+\frac{1}{w-\zeta}\right)\mathrm{d}\zeta.$$

Since $w=R^2/\overline{z}$ and $|\zeta|=R$, we have

$$\frac{1}{w-\zeta}=\frac{\overline{z}}{\zeta(\overline{\zeta}-\overline{z})}.$$

Therefore

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta-z}+\frac{\overline{z}}{\zeta(\overline{\zeta}-\overline{z})}\right)\mathrm{d}\zeta=\frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\frac{\zeta\overline{\zeta}-z\overline{z}}{\zeta(\zeta-z)(\overline{\zeta}-\overline{z})}\mathrm{d}\zeta.$$

Since we have

$$\operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right)=\frac{1}{2}\left(\frac{\zeta+z}{\zeta-z}+\frac{\overline{\zeta}+\overline{z}}{\overline{\zeta}-\overline{z}}\right)=\frac{\zeta\overline{\zeta}-z\overline{z}}{(\zeta-z)(\overline{\zeta}-\overline{z})}.$$

Substituting this into the previous formula gives the desired identity.

(b) Let $\zeta=R\mathrm{e}^{i\gamma}$ and $z=r$. Then the identity we just obtained in (a) becomes

$$\operatorname{Re}\left(\frac{R\mathrm{e}^{i\gamma}+r}{R\mathrm{e}^{i\gamma}-r}\right)=\frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

Assignment 3.7 (Exercise 2.6.12)

Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\Delta u(x,y)=0$$

for all $(x,y)\in\mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\operatorname{Re}(f)=u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z)=2\partial u/\partial z$. Therefore, let $g(z)=2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F'=g$? Prove that $\operatorname{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy's integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z=r\mathrm{e}^{i\theta}$ one has

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-\varphi)u(\mathrm{e}^{i\varphi})\mathrm{d}\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma)=\frac{1-r^2}{1-2r\cos\gamma+r^2}.$$

Proof (a) Define $g(z)=2\partial u/\partial z=u_x-iu_y$. Since $u\in C^2(\mathbb{D})$ and $\Delta u=0$, we have

$$\frac{\partial g}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u_x-iu_y)=\frac{1}{2}(u_{xx}+u_{yy})=0.$$

Hence g is holomorphic on $\mathbb{D}$. Thus by Theorem 2.1 of the textbook, g has a primitive F on $\mathbb{D}$ with

$F' = g$. Then $\operatorname{Re}(F)_x = \operatorname{Re}(F') = u_x$ and $\operatorname{Re}(F)_y = -\operatorname{Im}(F') = u_y$, hence $\operatorname{Re}(F) - u$ is constant. Thus there exists a real constant c such that $\operatorname{Re}(F-c) = u$. Letting $f = F-c$, we obtain a holomorphic function on $\mathbb{D}$ with real part u. If f_1 and f_2 are two such functions, then $\operatorname{Re}(f_1 - f_2) = 0$. Hence $f_1 - f_2$ is constant because it's holomorphic. Since its real part is 0, this constant is purely imaginary. Therefore the imaginary part of f is uniquely defined up to an additive real constant.
(b) By (a), let f be a holomorphic function on $\mathbb{D}$ with $\operatorname{Re}(f) = u$, and let $z = r\mathrm{e}^{i\theta}$. Applying Exercise 11(a) with $R_0 = 1$, for $|z| = r < 1$, choose R such that $0 \leqslant r < R < 1$, we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(R\mathrm{e}^{i\varphi}) \operatorname{Re}\left(\frac{R\mathrm{e}^{i\varphi} + z}{R\mathrm{e}^{i\varphi} - z}\right) \mathrm{d}\varphi.$$

Take real parts and let $R \to 1^-$, since u is continuous on $\overline{\mathbb{D}}$ and $|z| = r < 1$ is fixed, the integral converges uniformly in φ as $R \to 1^-$. Hence we can pass to the limit under the integral sign. So

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{e}^{i\varphi}) \operatorname{Re}\left(\frac{\mathrm{e}^{i\varphi} + r\mathrm{e}^{i\theta}}{\mathrm{e}^{i\varphi} - r\mathrm{e}^{i\theta}}\right) \mathrm{d}\varphi.$$

By Exercise 11(b),

$$\operatorname{Re}\left(\frac{\mathrm{e}^{i\varphi} + r\mathrm{e}^{i\theta}}{\mathrm{e}^{i\varphi} - r\mathrm{e}^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos(\varphi - \theta) + r^2} = P_r(\theta - \varphi).$$

Therefore we can obtain

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi) u(\mathrm{e}^{i\varphi}) \mathrm{d}\varphi.$$

Thus the Poisson representation integral formula holds for $z = re^{i\theta} \in \mathbb{D}$. And the boundary values $u(\mathrm{e}^{i\theta})$ are obtained by letting $r \to 1^-$, since the Poisson kernel forms an approximate identity and u is continuous on $\overline{\mathbb{D}}$. So we are done.

Chapter 4 Homework 4

Assignment 4.1 (Exercise 2.6.14)

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Proof Let the pole of f at z_0 have order m. Since f is holomorphic in a neighborhood of $\overline{\mathbb{D}} \setminus \{z_0\}$, we can write

$$f(z) = \sum_{k=1}^{m} \frac{c_k}{(1 - z/z_0)^k} + g(z),$$

where $c_m \neq 0$ and g is holomorphic in some disc $|z| < \rho$ with $\rho > 1$. For each $k \geqslant 1$ we have

$$\frac{1}{(1 - z/z_0)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z_0^{-n} z^n$$

for $|z| < |z_0| = 1$. Also, since g is holomorphic on $|z| < \rho$, we can write

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

with $|b_n| \leqslant Cr^{-n}$ for some $C > 0, 1 < r < \rho$ (by Cauchy inequalities). Therefore we have $a_n = z_0^{-n} P(n) + b_n$, where

$$P(n) := \sum_{k=1}^{m} c_k \binom{n+k-1}{k-1}.$$

This is a polynomial in n of degree $m-1$, and its leading coefficient is $c_m/(m-1)! \neq 0$. Therefore $P(n) \neq 0$ for all sufficiently large n, and

$$\frac{P(n)}{P(n+1)} \to 1\,(n \to \infty).$$

Since $|z_0| = 1$ and $r > 1$, we have $z_0^{-n} b_n \to 0$. Thus $a_n = z_0^{-n}(P(n) + o(1)), a_{n+1} = z_0^{-(n+1)}(P(n+1) + o(1))$. It follows that

$$\frac{a_n}{a_{n+1}} = z_0 \frac{P(n) + o(1)}{P(n+1) + o(1)} \to z_0\,(n \to \infty) \text{ i.e., } \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Assignment 4.2 (Exercise 2.6.15)

Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \text{ whenever } |z| = 1,$$

then f is constant.

[Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

Proof Consider the function

$$g(z) := \begin{cases} f(z), & |z| < 1 \\ 1/\overline{f(1/\overline{z})}, & |z| \geqslant 1 \end{cases}$$

If $|z| = 1$, then $1/\overline{z} = z$, and since $|f(z)| = 1$ on the unit circle, we have

$$\frac{1}{\overline{f(1/\overline{z})}} = \frac{1}{\overline{f(z)}} = f(z).$$

Hence g is continuous on $\mathbb{C}$. Now $1/f(1/z)$ is holomorphic on $|z| > 1$, since f is holomorphic and non-vanishing in $\mathbb{D}$. Fix z_0 with $|z_0| > 1$. Since $1/f(1/z)$ is holomorphic near $\overline{z_0}$, we can write

$$\frac{1}{f(1/z)} = \sum_{n=0}^{\infty} a_n (z - \overline{z}_0)^n \Rightarrow \frac{1}{\overline{f(1/\overline{z})}} = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n$$

near $z = z_0$. Thus g is holomorphic when $|z| > 1$. So g is holomorphic in $|z| < 1$ and in $|z| > 1$, and continuous on all of $\mathbb{C}$. Similar with the proof of the Schwarz reflection principle, using Morera's theorem and Cauchy's theorem, we can obtain g is holomorphic on $\mathbb{C}$. Since g is entire and bounded, Liouville's theorem implies that g is constant. Therefore f is constant as well.

Assignment 4.3 (Problem 2.7.1)

Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be regular for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \text{ for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$, then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.]

(b) Fix $0 < \alpha < \infty$. Show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \text{ for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

Proof (a) Let $\theta = 2\pi p/2^k, p, k \in \mathbb{N}^+$ and $z = re^{i\theta}, 0 \leqslant r < 1$. Then for $n \geqslant k$ we have $e^{i2^n\theta} = 1$, so

$$f(re^{i\theta}) = \sum_{n=0}^{k-1} r^{2^n} e^{i2^n\theta} + \sum_{n=k}^{\infty} r^{2^n} \Rightarrow |f(re^{i\theta})| \geqslant \sum_{n=k}^{\infty} r^{2^n} - \sum_{n=0}^{k-1} r^{2^n}.$$

As $r \to 1^-$, the first sum tends to $+\infty$, while the second remains bounded. Therefore $|f(re^{i\theta})| \to \infty$. Now the set of points $e^{i\theta}$ with $\theta = 2\pi p/2^k, p, k \in \mathbb{N}^+$ is dense on the unit circle. If f had an analytic continuation across some boundary point w, then there would exist an arc of the unit circle containing w on which f remained bounded near the boundary. But every such arc contains $e^{i2\pi p/2^k}$ for some $p, k \in \mathbb{N}^+$, and along the corresponding radius we have just proved that $|f(re^{i\theta})| \to \infty$ as $r \to 1^-$. This is impossible. Hence no point of the unit circle is regular for f, so f cannot be analytically continued past the unit circle.

(b) Since $\sum_{n=0}^{\infty} 2^{-n\alpha} < \infty$, the series $\sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$ converges uniformly on $\overline{\mathbb{D}}$. Thus f extends continuously to the unit circle. We now show that f cannot be analytically continued past the unit circle. Fix an integer $m > \alpha$. Let

$$L := z\frac{\mathrm{d}}{\mathrm{d}z}.$$

Then for $|z| < 1$, we have

$$L^m f(z) = \sum_{n=0}^{\infty} 2^{nm} 2^{-n\alpha} z^{2^n} = \sum_{n=0}^{\infty} 2^{n(m-\alpha)} z^{2^n}.$$

Suppose $\theta = 2\pi p/2^k$. Then for $n \geqslant k$ we have $e^{i2^n\theta} = 1$, so for $z = re^{i\theta}, 0 \leqslant r < 1$, we have

$$L^m f(re^{i\theta}) = \sum_{n=0}^{k-1} 2^{n(m-\alpha)} r^{2^n} e^{i2^n\theta} + \sum_{n=k}^{\infty} 2^{n(m-\alpha)} r^{2^n}.$$

The second sum consists of positive terms and tends to $+\infty$ as $r \to 1^-$, since $\sum_{n=k}^{\infty} 2^{n(m-\alpha)} = \infty$. Hence $|L^m f(re^{i\theta})| \to \infty$ as $r \to 1^-$. Again the set of points $\{e^{i2\pi p/2^k} : p, k \in \mathbb{N}^+\}$ is dense on the unit circle. If f could be analytically continued across some boundary point, then so could $L^m f$, and hence $L^m f$ would remain bounded near this point, which leads to a contradiction. Therefore f cannot be analytically continued past the unit circle.

Remark There is another way to prove that f cannot be analytically continued past the unit circle. Since $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$ converges uniformly on $\overline{\mathbb{D}}$, its boundary values define a continuous function $F(\theta) := f(e^{i\theta}) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n\theta}$. The real and imaginary parts of F are lacunary Fourier series of Weierstrass type and nowhere differentiable. But if f could be analytically continued across some boundary point, then $F(\theta)$ would be real analytic in a neighborhood of the corresponding angle. This is impossible. Hence f cannot be analytically continued past the unit circle.

Assignment 4.4 (Problem 2.7.2)

Let

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n \textit{ for } |z| < 1$$

where $d(n)$ denotes the number of divisors of n. Observe that the radius of convergence of this *series is* 1. *Verify the identity*

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}.$$

Using this identity, show that if $z = r$ *with* $0 < r < 1$, *then*

$$|F(r)| \geqslant c \frac{1}{1-r} \log \frac{1}{1-r}$$

as $r \to 1$. *Similarly, if* $\theta = 2\pi p/q$ *where* p *and* q *are positive integers and* $z = r\mathrm{e}^{i\theta}$, *then*

$$|F(r\mathrm{e}^{i\theta})| \geqslant c_{p/q} \frac{1}{1-r} \log \frac{1}{1-r}$$

as $r \to 1$. *Conclude that* F *cannot be continued analytically past the unit disc.*

Proof We first verify the identity. For $|z| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z^{mn}.$$

We can obtain that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |z^{mn}| = \sum_{n=1}^{\infty} \frac{|z|^n}{1-|z|^n} \leqslant \sum_{n=1}^{\infty} \frac{|z|^n}{1-|z|} = \frac{|z|}{(1-|z|)^2} < \infty.$$

So the double series converges absolutely, so we may rearrange it by the powers of z. The coefficient of z^N is exactly the number of positive divisors of N, namely $d(N)$. Hence

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{N=1}^{\infty} d(N) z^N = F(z).$$

Now if $z = r \in (0,1)$. Since all terms are positive and we have $1 - r^n = (1-r)(1+r+\cdots+r^{n-1}) \leqslant n(1-r)$, hence we can obtain

$$|F(r)| = F(r) = \sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \geqslant \sum_{n=1}^{\infty} \frac{r^n}{n(1-r)}. = -\frac{1}{1-r} \log(1-r) = \frac{1}{1-r} \log \frac{1}{1-r}.$$

Thus the desired estimate holds for all $0 < r < 1$ with $c = 1$. Now let $\theta = 2\pi p/q$, where $p, q \in \mathbb{N}^+$ and $z = r\mathrm{e}^{i\theta}$. Assume that $q > 1$ and $(p,q) = 1$ WLOG, by the identity above, we have

$$F(r\mathrm{e}^{i\theta}) = \sum_{n=1}^{\infty} \frac{r^n \mathrm{e}^{in\theta}}{1 - r^n \mathrm{e}^{in\theta}}.$$

We split the series according to the residue class of n modulo q, i.e., $F(r\mathrm{e}^{i\theta}) = A(r) + B(r)$, where

$$A(r) := \sum_{m=1}^{\infty} \frac{r^{mq}}{1-r^{mq}}, B(r) := \sum_{a=1}^{q-1} \sum_{m=0}^{\infty} \frac{r^{mq+a} \mathrm{e}^{ia\theta}}{1 - r^{mq+a} \mathrm{e}^{ia\theta}}.$$

By the estimate already proved for real arguments, we have

$$A(r) = \sum_{m=1}^{\infty} \frac{(r^q)^m}{1-(r^q)^m} \geqslant \frac{1}{1-r^q} \log \frac{1}{1-r^q}.$$

Also, since $1 - r^q = (1-r)(1+r+\cdots+r^{q-1}) \leqslant q(1-r)$, we have

$$\frac{1}{1-r^q} \geqslant \frac{1}{q(1-r)}, \log \frac{1}{1-r^q} \geqslant \log \frac{1}{q(1-r)} = \log \frac{1}{1-r} - \log q.$$

Hence there exists $r_0 \in (0,1)$ such that for $r_0 < r < 1$,

$$\log \frac{1}{1-r^q} \geqslant \frac{1}{2} \log \frac{1}{1-r}.$$

Therefore, for $r_0 < r < 1$,

$$A(r) \geqslant \frac{1}{2q} \frac{1}{1-r} \log \frac{1}{1-r}.$$

It remains to estimate $B(r)$. For $1 \leqslant a \leqslant q-1$, we have $\mathrm{e}^{ia\theta} \neq 1$. Let $\delta(p/q) := \min_{1 \leqslant a \leqslant q-1} |1 - \mathrm{e}^{ia\theta}| > 0$. If r is sufficiently close to 1, then for every $m \in \mathbb{N}$ and every $1 \leqslant a \leqslant q-1$, we have

$$|1 - r^{mq+a}\mathrm{e}^{ia\theta}| \geqslant |1 - \mathrm{e}^{ia\theta}| - |(1 - r^{mq+a})\mathrm{e}^{ia\theta}| \geqslant \delta - (1 - r^{mq+a}) \geqslant \frac{\delta}{2}.$$

Thus for r sufficiently close to 1, we have

$$\left| \frac{r^{mq+a}\mathrm{e}^{ia\theta}}{1 - r^{mq+a}\mathrm{e}^{ia\theta}} \right| \leqslant \frac{2}{\delta} r^{mq+a}.$$

Hence

$$|B(r)| \leqslant \frac{2}{\delta} \sum_{a=1}^{q-1} \sum_{m=0}^{\infty} r^{mq+a} \leqslant \frac{2(q-1)}{\delta} \sum_{m=0}^{\infty} r^{mq} = \frac{2(q-1)}{\delta} \frac{1}{1-r^q} \leqslant \frac{2(q-1)}{\delta} \frac{1}{1-r}.$$

So there exists a constant $C_{p/q} > 0$ such that $|B(r)| \leqslant C_{p/q}/(1-r)$ for all r sufficiently close to 1. Now we can obtain

$$|F(r\mathrm{e}^{i\theta})| \geqslant A(r) - |B(r)| \geqslant \frac{1}{2q} \frac{1}{1-r} \log \frac{1}{1-r} - C_{p/q} \frac{1}{1-r}.$$

Since $\log \frac{1}{1-r} \to +\infty$ as $r \to 1^-$, there exists $r_1 \in (0,1)$ such that for $r_1 < r < 1$, we have

$$C_{p/q} \leqslant \frac{1}{4q} \log \frac{1}{1-r}.$$

Therefore, we can obtain

$$|F(r\mathrm{e}^{i\theta})| \geqslant \frac{1}{4q} \frac{1}{1-r} \log \frac{1}{1-r} \triangleq c_{p/q} \frac{1}{1-r} \log \frac{1}{1-r}$$

as $r \to 1^-$ for some constant $c_{p/q} > 0$ depending only on p/q. Since the set of points $\{\mathrm{e}^{2\pi i p/q} : p, q \in \mathbb{N}^+, (p,q) = 1\}$ is dense on the unit circle, no point of the unit circle can be regular for F. Hence F cannot be continued analytically past the unit disc.

Assignment 4.5 (Problem 2.7.3)

Morera's theorem states that if f is continuous in $\mathbb{C}$, and $\int_T f(z)\mathrm{d}z = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.

(a) Suppose that f is continuous on $\mathbb{C}$, and

$$(*) \qquad \int_C f(z)\mathrm{d}z = 0$$

for every circle C. Prove that f is holomorphic.

(b) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)\mathrm{d}z = 0 \text{ for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z)\mathrm{d}z = 0$ for all equilateral triangles.

[Hint: As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b(\overline{z} - \overline{z_0}) + O(|z - z_0|^2)$ for z near z_0. Integrating this expansion over small circles around z_0 yields $\partial f/\partial\overline{z} = b = 0$ at z_0. Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

In general, let $\varphi(w) = \varphi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leqslant \varphi(w) \leqslant 1$, and $\int_{\mathbb{R}^2} \varphi(w)\mathrm{d}V(w) = 1$, where $\mathrm{d}V(w) = \mathrm{d}x\mathrm{d}y$, and $\int$ denotes the usual integral of a function of two variables in $\mathbb{R}^2$. For each $\varepsilon > 0$, let $\varphi_\varepsilon(z) = \varepsilon^{-2}\varphi(\varepsilon^{-1}z)$, as well as

$$f_\varepsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\varepsilon(w)\mathrm{d}V(w),$$

where the integral denotes the usual integral of functions of two variables, with $\mathrm{d}V(w)$ the area element of $\mathbb{R}^2$. Then f_ε is smooth, satisfies condition $(*)$, and $f_\varepsilon \to f$ uniformly on any compact subset of $\mathbb{C}$.]

Proof We use the mollifier indicated in the hint. Define that

$$\varphi : \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{R}, z \mapsto \begin{cases} c\exp\left(-\frac{1}{(|z|^2-1/4)^2}\right), & |z| < 1/2 \\ 0, & |z| \geqslant 1/2 \end{cases}$$

where $c > 0$ is chosen such that $\int_{\mathbb{R}^2} \varphi(w)\mathrm{d}V(w) = 1$. For each $\varepsilon > 0$, define

$$\varphi_\varepsilon(z) = \varepsilon^{-2}\varphi(\varepsilon^{-1}z),\ f_\varepsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\varepsilon(w)\mathrm{d}V(w).$$

Then f_ε is smooth and $f_\varepsilon \to f$ uniformly on every compact subset of $\mathbb{C}$ as $\varepsilon \to 0$. Moreover, for part (a), if C is a circle, then $C - w$ is again a circle, thus we have

$$\int_C f_\varepsilon(z)\mathrm{d}z = \int_{\mathbb{R}^2}\left(\int_C f(z - w)\mathrm{d}z\right)\varphi_\varepsilon(w)\mathrm{d}V(w) = 0.$$

(Since φ_ε has compact support and f is continuous, so we can change the order of integration.) For part (b), if $\gamma \in \mathcal{F}$, then for every w the translate $\gamma - w$ still belongs to $\mathcal{F}$, so similarly we have

$$\int_\gamma f_\varepsilon(z)\mathrm{d}z = \int_{\mathbb{R}^2}\left(\int_\gamma f(z - w)\mathrm{d}z\right)\varphi_\varepsilon(w)\mathrm{d}V(w) = 0.$$

Thus, by Theorem 5.2 of the textbook, it suffices to prove the result when f is smooth.

(a) Fix $z_0 \in \mathbb{C}$. Since f is twice real differentiable, we can write

$$f(z) = f(z_0) + a(z - z_0) + b(\overline{z} - \overline{z_0}) + O(|z - z_0|^2)$$

for z near z_0. Integrating over the circle $|z - z_0| = \varepsilon$ for sufficiently small $\varepsilon > 0$, we have

$$0 = \int_{|z-z_0|=\varepsilon} f(z)\mathrm{d}z = \int_{|z-z_0|=\varepsilon} [b(\overline{z} - \overline{z_0}) + O(|z - z_0|^2)]\mathrm{d}z.$$

$$\Rightarrow 0 = \int_0^{2\pi} b\varepsilon\mathrm{e}^{-i\theta}i\varepsilon\mathrm{e}^{i\theta}\mathrm{d}\theta + O(\varepsilon^3) = 2\pi ib\varepsilon^2 + O(\varepsilon^3).$$

Then we have $2\pi i b + O(\varepsilon) = 0$, letting $\varepsilon \to 0$, we can obtain $b = 0$. that is $\partial f/\partial \overline{z}(z_0) = 0$. Since z_0 is arbitrary, f is holomorphic.
(b) Fix $z_0 \in \mathbb{C}$. Since f is twice real differentiable, again write

$$f(z) = f(z_0) + a(z - z_0) + b(\overline{z} - \overline{z_0}) + O(|z - z_0|^2)$$

for z near z_0. Let $\Gamma_\varepsilon := z_0 + \varepsilon\Gamma$, which belongs to $\mathcal{F}$. Then

$$0 = \int_{\Gamma_\varepsilon} f(z)\mathrm{d}z = \int_{\Gamma_\varepsilon} [b(\overline{z} - \overline{z_0}) + O(|z - z_0|^2)]\mathrm{d}z.$$

Writing $z = z_0 + \varepsilon w$ with $w \in \Gamma$, we get

$$0 = b\varepsilon^2 \int_\Gamma \overline{w}\mathrm{d}w + O(\varepsilon^3).$$

Since Γ forms a region of positive area, let $w = x + iy$, by Green's theorem, we have

$$\int_\Gamma \overline{w}\mathrm{d}w = \int_\Gamma x\mathrm{d}x + y\mathrm{d}y + i\int_\Gamma x\mathrm{d}y - y\mathrm{d}x = 2i \cdot \mathrm{Area}(\Omega_\Gamma) \neq 0.$$

Therefore $b = 0$, that is $\partial f/\partial \overline{z}(z_0) = 0$. Since z_0 is arbitrary, f is holomorphic. Taking Γ to be an equilateral triangle gives the last statement.

Assignment 4.6 (Problem 2.7.4)

Prove the converse to Runge's theorem: if K is a compact set whose complement is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomials on K.
[Hint: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (Chapter 3) to show that this inequality continues to hold for all z in the component of K^c that contains z_0.]

Proof Since K is compact and K^c is not connected, so K^c has a bounded component, call it U, and pick a point $z_0 \in U$. Define $f(z) = 1/(z - z_0)$. Then f is holomorphic in a neighborhood of K, since $z_0 \notin K$ and K is compact. Suppose, for contradiction, that f can be approximated uniformly by polynomials on K. Let $M := \max_{z\in K} |z - z_0| + 1$. Then there exists a polynomial p such that

$$|f(z) - p(z)| < \frac{1}{M} \Rightarrow |(z - z_0)p(z) - 1| < \frac{|z - z_0|}{M} < 1$$

for all $z \in K$. Define $h(z) := (z - z_0)p(z) - 1$, then $h(z)$ is entire. Since we have $\partial U \subset K$, so $|h(z)| < 1$ on ∂U. By the maximum modulus principle, $|h(z)| < 1$ on U. In particular, taking $z = z_0$ gives $|h(z_0)| = |-1| = 1 < 1$, which leads to a contradiction. Therefore f cannot be approximated uniformly by polynomials on K. So we are done.

Assignment 4.7 (Problem 2.7.5)

There exists an entire function F with the following "universal" property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^\infty$ of positive integers, so that

$$\lim_{k\to\infty} F(z + N_k) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$.

(a) Let $p_1, p_2, \cdots$ denote an enumeration of the collection of polynomials whose coefficients have rational real and imaginary parts. Show that it suffices to find an entire function F and an increasing sequence $\{M_n\}$ of positive integers, such that

$$(\star) \quad |F(z) - p_n(z - M_n)| < \frac{1}{n} \text{ whenever } z \in D_n,$$

where D_n denotes the disc centered at M_n and of radius n. [Hint: Given h entire, there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} p_{n_k}(z) = h(z)$ uniformly on every compact subset of $\mathbb{C}$.]
(b) Construct F satisfying $(\star)$ as an infinite series

$$F(z) = \sum_{n=1}^{\infty} u_n(z)$$

where $u_n(z) = p_n(z - M_n)\mathrm{e}^{-c_n(z-M_n)^2}$, and the quantities $c_n > 0$ and $M_n > 0$ are chosen appropriately with $c_n \to 0$ and $M_n \to \infty$. [Hint: The function e^{-z^2} vanishes rapidly as $|z| \to \infty$ in the sectors $\{|\arg z| < \pi/4 - \delta\}$ and $\{|\pi - \arg z| < \pi/4 - \delta\}$.]

In the same spirit, there exists an alternate "universal" entire function G with the following property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers, so that

$$\lim_{k\to\infty} D^{N_k} G(z) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$. Here $D^j G$ denotes the j^{th} (complex) derivative of G. ♠

Proof (a) Assume that F and $\{M_n\}$ satisfy $(\star)$. Let h be any entire function. Since polynomials with rational real and imaginary coefficients are dense in the space of entire functions for the topology of uniform convergence on compact sets, there exists a subsequence $\{p_{n_k}\}$ such that $p_{n_k}(z) \to h(z)$ uniformly on every compact subset of $\mathbb{C}$. We claim that $N_k := M_{n_k}$ works. Let $K \subset \mathbb{C}$ be compact. For k large enough, we have $K + M_{n_k} \subset D_{n_k}$. Hence for $z \in K$ and sufficiently large k, we have

$$|F(z + N_k) - h(z)| \leqslant |F(z + M_{n_k}) - p_{n_k}(z)| + |p_{n_k}(z) - h(z)| < \frac{1}{n_k} + |p_{n_k}(z) - h(z)|.$$

The RHS tends to 0 uniformly on K. Therefore $F(z + N_k) \to h(z)$ uniformly on every compact subset of $\mathbb{C}$. So it suffices to find F and $\{M_n\}$ satisfying $(\star)$.
(b) We construct $u_n(z) = p_n(z - M_n)\mathrm{e}^{-c_n(z-M_n)^2}$ recursively. For each fixed $n \in \mathbb{N}^+$, choose $c_n > 0$ sufficiently small such that $c_n \to 0$ as $n \to \infty$, and also such that

$$|p_n(w)(\mathrm{e}^{-c_n w^2} - 1)| < \frac{1}{3n} \text{ for all } |w| \leqslant n. \tag{1}$$

This is possible because for fixed $n \in \mathbb{N}^+$, $\mathrm{e}^{-cw^2} \to 1$ uniformly on $|w| \leqslant n$ as $c \to 0$. Now choose any $M_1 \in \mathbb{N}^+$, then suppose $u_1, \cdots, u_{n-1}$ have been chosen. Let $B_{n-1} := B(0, M_{n-1} + n - 1)$ $(n \geqslant 2)$. Since each factor $\mathrm{e}^{-c_j(z-M_j)^2}, 1 \leqslant j \leqslant n-1$ decays faster than any polynomial as $z \to \infty$ in the sector near the positive or negative real axes, we can choose $M_n \in \mathbb{N}^+$ so large that $D_n \cap B_{n-1} = \varnothing$ and

$$\sum_{j=1}^{n-1} |u_j(z)| < \frac{1}{3n} \text{ for } z \in D_n,\ |u_n(z)| < \frac{1}{2^{n+2}} \text{ for } z \in B_{n-1}. \tag{2}$$

Indeed, when $z \in D_n$, each $z - M_j$ for $1 \leqslant j \leqslant n-1$ lies far out in a sector near the positive real axis, so $|u_j(z)|$ is tiny there; when $z \in B_{n-1}$, $z - M_n$ lies far out in a sector near the negative real axis, so $|u_n(z)|$ is tiny there. Also, we can obtain $M_n \uparrow +\infty$ as $n \to \infty$. Define

$$F(z) := \sum_{n=1}^{\infty} u_n(z).$$

We first show that the series converges uniformly on every compact set. Let K be compact. Since K is bounded, there exists $N \in \mathbb{N}^+$ such that $K \subset B_N$. Then by (2), for all $m > N$, we have

$$|u_m(z)| < \frac{1}{2^{m+2}} \text{ for all } z \in K$$

Therefore the series converges uniformly on K. Since K is arbitrary, the series converges uniformly on every compact subset of $\mathbb{C}$. Thus F is entire by Theorem 5.2 of the textbook. It remains to prove $(*)$. Fix n and let $z \in D_n$. Then

$$|F(z) - p_n(z - M_n)| \leqslant \sum_{j=1}^{n-1} |u_j(z)| + |u_n(z) - p_n(z - M_n)| + \sum_{m=n+1}^{\infty} |u_m(z)|.$$

By (2), the first term is $< 1/(3n)$. By (1), the middle term is also $< 1/(3n)$. As for $m > n$, we have $D_n \subset B_{m-1}$, so by (2) again, we have $|u_m(z)| < 2^{-m-2}$. Hence

$$\sum_{m=n+1}^{\infty} |u_m(z)| < \sum_{m=n+1}^{\infty} \frac{1}{2^{m+2}} = \frac{1}{2^{n+2}} < \frac{1}{3n}.$$

Combining these estimates, we obtain

$$|F(z) - p_n(z - M_n)| < \frac{1}{n}$$

for all $z \in D_n$. This is exactly $(*)$. So we are done.

We now prove the last statement. We can write

$$p_n(z) = \sum_{j=0}^{d_n} a_{n,j} z^j.$$

We should construct an entire function G and a strictly increasing sequence of positive integers $\{N_n\}$ such that, for each $n \in \mathbb{N}^+$, we can obtain

$$|D^{N_n} G(z) - p_n(z)| < \frac{1}{n} \text{ for } |z| \leqslant n. \tag{†}$$

Once this is done, the desired universality follows exactly as in part (a): given any entire function h, choose a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to h$ uniformly on every compact subset of $\mathbb{C}$. Then for every compact set $K \subset \mathbb{C}$, if $K \subset \{z : |z| \leqslant R\}$ and $n_k \geqslant R$, we have

$$|D^{N_{n_k}} G(z) - h(z)| \leqslant |D^{N_{n_k}} G(z) - p_{n_k}(z)| + |p_{n_k}(z) - h(z)| < \frac{1}{n_k} + |p_{n_k}(z) - h(z)|$$

for all $z \in K$, and the RHS tends to 0 uniformly on K. To construct G, we choose $\{N_n\}$ recursively. We choose any $N_1 \in \mathbb{N}^+$. For each n, define

$$q_n(z) := \sum_{j=0}^{d_n} a_{n,j} \frac{j!}{(N_n + j)!} z^{N_n + j} \Rightarrow D^{N_n} q_n(z) = p_n(z).$$

Also, if $m < n$ and $N_n > N_m + d_m$, then $D^{N_n} q_m(z) = 0$. So we choose $N_n \in \mathbb{N}^+$ so large that

$$N_n > N_{n-1} + d_{n-1} \, (n \geqslant 2), \; |q_n(z)| < 2^{-n} \text{ for } |z| \leqslant n. \tag{3}$$

and moreover

$$|D^{N_m} q_n(z)| < 2^{-n} \text{ for } |z| \leqslant m, 1 \leqslant m < n. \tag{4}$$

This is possible because for fixed n and fixed $1 \leqslant m < n$, we have

$$D^{N_m} q_n(z) = \sum_{j=0}^{d_n} a_{n,j} \frac{j!}{(N_n + j - N_m)!} z^{N_n + j - N_m}.$$

So on the compact set $|z| \leqslant m$, we have

$$\sup_{|z| \leqslant m} |D^{N_m} q_n(z)| \leqslant \sum_{j=0}^{d_n} |a_{n,j}| \cdot d_n! \frac{m^{N_n + j - N_m}}{(N_n + j - N_m)!} \to 0$$

as $N_n \to \infty$. The process of obtaining (3) is similar. Now define

$$G(z) := \sum_{n=1}^{\infty} q_n(z).$$

We claim that this series converges uniformly on every compact subset of $\mathbb{C}$. Let $K \subset \mathbb{C}$ be compact. $\exists R > 0$ such that $K \subset \{z : |z| \leqslant R\}$. Then by (3), for all $n \geqslant R$ we have $|q_n(z)| < 2^{-n}$ for $z \in K$. Hence the series converges uniformly on K. Therefore G is entire by Theorem 5.2 of the textbook. We next prove $(\dagger)$. Fix n and let $|z| \leqslant n$. We have $D^{N_n} q_m = 0$ for all $1 \leqslant m < n$. Hence

$$D^{N_n} G(z) - p_n(z) = \sum_{m=n+1}^{\infty} D^{N_n} q_m(z).$$

For every $m > n \geqslant 1$, since $|z| \leqslant n$, so by (4) we have $|D^{N_n} q_m(z)| < 2^{-m}$. Therefore

$$|D^{N_n} G(z) - p_n(z)| \leqslant \sum_{m=n+1}^{\infty} 2^{-m} = \frac{1}{2^n} < \frac{1}{n}.$$

Thus $(\dagger)$ holds. This proves the last statement.

Chapter 5 Homework 5

Assignment 5.1

Let $\Pi = \{u = 0\} \cong \mathbb{C}$ be the horizontal plane in $\mathbb{R}^3$, where (x, y, u) are the standard coordinates. Consider the two spheres

$$S = \{x^2 + y^2 + u^2 = 1\} \text{ and } T = \left\{x^2 + y^2 + \left(u - \frac{1}{2}\right)^2 = \frac{1}{4}\right\}.$$

*Both spheres have the same north pole $N = (0, 0, 1)$. For each sphere, define **stereographic projection** from N onto Π: for a point on the sphere different from N, draw the line through N and that point, and let its intersection with Π be the image. Denote these projections by $\sigma : S \setminus \{N\} \to \Pi$, $\tau : T \setminus \{N\} \to \Pi$.*

(a) Give the explicit formulas for σ and τ.

(b) Show that, on $S \setminus \{N, (0, 0, -1)\}$, the stereographic images of the two points (x, y, u) and $(x, y, -u)$ are inverse points with respect to the unit circle.

(c) Show that, on $T \setminus \{N, O\}$, the stereographic images of the two points (x, y, u) and $(x, y, 1-u)$ are inverse points with respect to the unit circle.

Proof (a) Let $P = (x, y, u) \in S \setminus \{N\}$. The line through N and P is $(0, 0, 1) + t(x, y, u - 1)$. Its intersection with Π is obtained from $1+t(u-1) = 0$, hence $t = 1/(1-u)$ $(u \neq 1)$. Hence $\sigma(x, y, u) = (1 - u)^{-1}(x, y, 0)$. We identify it with the complex number, so $\sigma(x, y, u) = (x + iy)/(1 - u)$. Exactly the same computation gives, for $Q = (x, y, u) \in T \setminus \{N\}$, we have $\tau(x, y, u) = (1 - u)^{-1}(x, y, 0)$. That is, $\tau(x, y, u) = (x + iy)/(1 - u)$ with the complex number form.

(b) Let $P = (x, y, u) \in S \setminus \{N, (0, 0, -1)\}$ and $P' = (x, y, -u)$. Denote $z = \sigma(P)$ and $z' = \sigma(P')$, respectively. Then we have $z = (x + iy)/(1 - u)$ and $z' = (x + iy)/(1 + u)$ by (a). Since $P \in S$, we have $x^2 + y^2 + u^2 = 1$, so $x^2 + y^2 = 1 - u^2$. Therefore

$$|z|^2 = \frac{x^2 + y^2}{(1 - u)^2} = \frac{1 + u}{1 - u}, \quad \frac{1}{\overline{z}} = \frac{z}{|z|^2} = \frac{x + iy}{1 - u} \cdot \frac{1 - u}{1 + u} = \frac{x + iy}{1 + u} = z'.$$

(c) Let $Q = (x, y, u) \in T \setminus \{N, O\}$ and $Q' = (x, y, 1 - u)$. Denote $w = \tau(Q)$ and $w' = \tau(Q')$, respectively. Then we have $w = (x + iy)/(1 - u)$ and $w' = (x + iy)/u$ by (a). Since $Q \in T$, we have $x^2 + y^2 + (u - 1/2)^2 = 1/4$, so $x^2 + y^2 = u(1 - u)$. Therefore

$$|w|^2 = \frac{x^2 + y^2}{(1 - u)^2} = \frac{u}{1 - u}, \quad \frac{1}{\overline{w}} = \frac{w}{|w|^2} = \frac{x + iy}{1 - u} \cdot \frac{1 - u}{u} = \frac{x + iy}{u} = w'.$$

Assignment 5.2

Take $f(z) = \dfrac{e^{az}}{1 + e^z}$. Prove that f is not a meromorphic function on the extended complex plane $\widehat{\mathbb{C}}$ for any $a \in \mathbb{C}$.

Proof The poles of f are the zeros of $1 + e^z$, that is $e^z = -1$, so they are exactly the points $z = (2k + 1)\pi i, k \in \mathbb{Z}$. Since $(1 + e^z)' = e^z$, and $e^{(2k+1)\pi i} = -1 \neq 0$, each of these zeros is simple.

Therefore f has infinitely many simple poles at $(2k+1)\pi i$, $k \in \mathbb{Z}$. Now let $U_R = \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ be an arbitrary neighborhood of ∞ in $\widehat{\mathbb{C}}$. Since there exist infinitely many integers k such that $|(2k+1)\pi i| > R$, U_R contains infinitely many poles of f. And ∞ must be a limit point of these poles and $\infty \in U_R$. So f is not meromorphic at ∞. So for any $a \in \mathbb{C}$, f is not meromorphic on $\widehat{\mathbb{C}}$.

Assignment 5.3 (Exercise 3.8.1)

Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ *are exactly at the integers, and that they are each of order* 1. *Calculate the residue of* $1/\sin \pi z$ *at* $z = n \in \mathbb{Z}$.

Proof By Euler's formula, $\sin \pi z = 0$ if and only if $e^{i\pi z} = e^{-i\pi z}$, equivalently $e^{2\pi i z} = 1$. Writing $z = x + iy$, we have $e^{2\pi i z} = e^{2\pi i x - 2\pi y}$. Hence $e^{2\pi i z} = 1$ if and only if $y = 0$ and $x \in \mathbb{Z}$. Therefore the zeros of $\sin \pi z$ are exactly the integers. Now take $n \in \mathbb{Z}$. Since $(\sin \pi z)' = \pi \cos \pi z$, we have $(\sin \pi z)'|_{z=n} = \pi \cos \pi n = \pi(-1)^n \neq 0$. Thus each zero is of order 1. Since n is a simple zero of the function $\sin \pi z$, it is a simple pole of the function $1/\sin \pi z$, so we have

$$\operatorname{Res}\left(\frac{1}{\sin \pi z}, n\right) = \lim_{z \to n} \frac{z-n}{\sin \pi z} = \frac{1}{(\sin \pi z)'|_{z=n}} = \frac{1}{\pi \cos \pi n} = \frac{(-1)^n}{\pi}.$$

Assignment 5.4 (Exercise 3.8.2)

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $1/(1+z^4)$*?*

Solution *Let* $f(z) = 1/(1+z^4)$. *The poles are the zeros of* $1+z^4$, *namely the fourth roots of* -1: $\omega_k = e^{ik\pi/4}$, $k = 1, 3, 5, 7$. *They are all simple. The poles in the upper half-plane are* $\omega_1 = e^{i\pi/4}$ *and* $\omega_3 = e^{3i\pi/4}$. *For a simple pole* ω_k *of* f, *we have* $\operatorname{Res}(f, \omega_k) = 1/(4\omega_k^3) = -\omega_k/4$, *since* $\omega^4 = -1$. *Therefore we have*

$$\operatorname{Res}(f, \omega_1) + \operatorname{Res}(f, \omega_3) = -\frac{\omega_1 + \omega_3}{4} = -\frac{1}{4}\left(e^{i\pi/4} + e^{3i\pi/4}\right) = -\frac{\sqrt{2}i}{4}.$$

Let $\gamma = \gamma_1 \cup \gamma_2$ *be the standard upper semicircular contour of radius* $R > 1$, *where* $\gamma_1 = [-R, R]$ *and* γ_2 *is the upper semicircle. By the residue formula, we have*

$$\int_{\gamma} f(z)dz = 2\pi i\left(\operatorname{Res}(f, \omega_1) + \operatorname{Res}(f, \omega_3)\right) = \frac{\sqrt{2}\pi}{2}.$$

On γ_2 *we write* $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. *Then for* $R > 1$, *we have*

$$\left|\int_{\gamma_2} f(z)dz\right| \leqslant \int_0^{\pi} \frac{Rd\theta}{|1+R^4e^{4i\theta}|} \leqslant \int_0^{\pi} \frac{Rd\theta}{R^4-1} = \frac{\pi R}{R^4-1} \to 0\,(R \to +\infty).$$

Therefore we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{\gamma} f(z)dz = \frac{\sqrt{2}\pi}{2}.$$

Assignment 5.5 (Exercise 3.8.3)

Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \mathrm{d}x = \pi \frac{\mathrm{e}^{-a}}{a}, \text{ for } a > 0.$$

Proof Let $f(z) = \mathrm{e}^{iz}/(z^2+a^2)$ and consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > a > 0$. The only pole in the upper half-plane is at $z = ai$, and it is simple. Thus

$$\operatorname{Res}(f, ai) = \lim_{z \to ai} \frac{\mathrm{e}^{iz}}{z+ai} = \frac{\mathrm{e}^{-a}}{2ai}.$$

Hence by the residue formula, we have

$$\int_{\gamma} f(z) \mathrm{d}z = 2\pi i \operatorname{Res}(f, ai) = \pi \frac{\mathrm{e}^{-a}}{a}.$$

On γ_2, write $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Then $|\mathrm{e}^{iz}| = \mathrm{e}^{-R\sin\theta}$, so for $R > a$, we have

$$\left| \int_{\gamma_2} f(z) \mathrm{d}z \right| \leqslant \int_0^{\pi} \frac{R\mathrm{e}^{-R\sin\theta}}{R^2-a^2} \mathrm{d}\theta = 2 \int_0^{\pi/2} \frac{R\mathrm{e}^{-R\sin\theta}}{R^2-a^2} \mathrm{d}\theta.$$

Using $\sin\theta \geqslant 2\theta/\pi$ on $[0, \pi/2]$, we can obtain

$$\left| \int_{\gamma_2} f(z) \mathrm{d}z \right| \leqslant \frac{2R}{R^2-a^2} \int_0^{\pi/2} \mathrm{e}^{-2R\theta/\pi} \mathrm{d}\theta = \frac{\pi(1-\mathrm{e}^{-R})}{R^2-a^2} \to 0 \, (R \to +\infty).$$

Therefore we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x^2+a^2} \mathrm{d}x = \pi \frac{\mathrm{e}^{-a}}{a}.$$

Taking the real part gives

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \mathrm{d}x = \pi \frac{\mathrm{e}^{-a}}{a}, \text{ for } a > 0.$$

Assignment 5.6 (Exercise 3.8.4)

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} \mathrm{d}x = \pi \mathrm{e}^{-a}, \text{ for all } a > 0.$$

Proof Let $f(z) = z\mathrm{e}^{iz}/(z^2+a^2)$ and consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > a > 0$. The only pole in the upper half-plane is at $z = ai$, and it is simple. Thus

$$\operatorname{Res}(f, ai) = \lim_{z \to ai} \frac{z\mathrm{e}^{iz}}{z+ai} = \frac{ai\mathrm{e}^{-a}}{2ai} = \frac{\mathrm{e}^{-a}}{2}.$$

Hence by the residue formula, we have

$$\int_{\gamma} f(z) \mathrm{d}z = 2\pi i \operatorname{Res}(f, ai) = i\pi\mathrm{e}^{-a}.$$

On γ_2, write $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Then $|\mathrm{e}^{iz}| = \mathrm{e}^{-R\sin\theta}$, so for $R > a$, we have

$$\left| \int_{\gamma_2} f(z) \mathrm{d}z \right| \leqslant \int_0^{\pi} \frac{R^2\mathrm{e}^{-R\sin\theta}}{R^2-a^2} \mathrm{d}\theta = 2 \int_0^{\pi/2} \frac{R^2\mathrm{e}^{-R\sin\theta}}{R^2-a^2} \mathrm{d}\theta.$$

Using $\sin\theta \geqslant 2\theta/\pi$ on $[0, \pi/2]$, we can obtain

$$\left| \int_{\gamma_2} f(z) \mathrm{d}z \right| \leqslant \frac{2R^2}{R^2-a^2} \int_0^{\pi/2} \mathrm{e}^{-2R\theta/\pi} \mathrm{d}\theta = \frac{\pi R(1-\mathrm{e}^{-R})}{R^2-a^2} \to 0 \, (R \to +\infty).$$

Therefore we have

$$\int_{-\infty}^{\infty} \frac{x\mathrm{e}^{ix}}{x^2+a^2}\mathrm{d}x = i\pi\mathrm{e}^{-a}.$$

Taking the imaginary part gives

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\mathrm{d}x = \pi\mathrm{e}^{-a}, \text{ for all } a > 0.$$

Assignment 5.7 (Exercise 3.8.5)

Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{(1+x^2)^2}\mathrm{d}x = \frac{\pi}{2}(1+2\pi|\xi|)\mathrm{e}^{-2\pi|\xi|}$$

for all ξ real.

Proof Since the integral is unchanged if we replace ξ by $-\xi$ and take complex conjugates, it suffices to prove the case $\xi \leqslant 0$. Let $f(z) = \mathrm{e}^{-2\pi i z\xi}/(1+z^2)^2$. Consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > 1$. The only pole in the upper half-plane is at $z = i$, and it is of order 2. By the residue formula, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i\,\mathrm{Res}(f,i) = 2\pi i \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-i)^2 f(z)\right) = 2\pi i \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{e}^{-2\pi i z\xi}}{(z+i)^2}\right)$$

$$= -2\pi i \lim_{z\to i}\frac{2\pi i\xi(z+i)\mathrm{e}^{-2\pi i z\xi}+2\mathrm{e}^{-2\pi i z\xi}}{(z+i)^3} = \frac{\pi}{2}(1-2\pi\xi)\mathrm{e}^{2\pi\xi}.$$

On γ_2, write $z = R\mathrm{e}^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Then $|\mathrm{e}^{-2\pi i z\xi}| = \mathrm{e}^{2\pi R\xi\sin\theta} \leqslant 1$ since $\xi \leqslant 0$. So for $R > 1$ we have

$$\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| \leqslant \int_0^\pi \frac{R\mathrm{d}\theta}{(R^2-1)^2} = \frac{\pi R}{(R^2-1)^2} \to 0\,(R\to+\infty).$$

Therefore we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{(1+x^2)^2}\mathrm{d}x = \frac{\pi}{2}(1-2\pi\xi)\mathrm{e}^{2\pi\xi}, \text{ for } \xi \leqslant 0.$$

Since for $\xi \leqslant 0$ we have $|\xi| = -\xi$, this becomes

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{(1+x^2)^2}\mathrm{d}x = \frac{\pi}{2}(1+2\pi|\xi|)\mathrm{e}^{-2\pi|\xi|}, \text{ for } \xi \leqslant 0.$$

Then the equality holds for all ξ real. So we are done.

Chapter 6 Homework 6

Assignment 6.1 (Exercise 3.8.6)

Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

Proof Let $f(z) = 1/(1+z^2)^{n+1}$ and consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > 1$. The only pole in the upper half-plane is at $z = i$, and it is of order $n+1$. Thus

$$\begin{aligned}\operatorname{Res}(f,i) &= \frac{1}{n!} \lim_{z\to i} \frac{\mathrm{d}^n}{\mathrm{d}z^n}((z-i)^{n+1} f(z)) = \frac{1}{n!} \lim_{z\to i} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{1}{(z+i)^{n+1}} \\ &= \frac{1}{n!} \cdot \frac{(-1)^n (2n)!}{n!(2i)^{2n+1}} = \frac{(2n)!}{2i\,(2^n n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{2i}.\end{aligned}$$

On γ_2 we write $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Then, for $R > 1$, we have

$$\left| \int_{\gamma_2} f(z)\mathrm{d}z \right| \leqslant \int_0^{\pi} \frac{R\mathrm{d}\theta}{(R^2-1)^{n+1}} = \frac{\pi R}{(R^2-1)^{n+1}} \to 0 \, (R \to +\infty).$$

Hence by the residue formula, we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{n+1}} = \lim_{R\to+\infty} \int_{\gamma} f(z)\mathrm{d}z = 2\pi i \operatorname{Res}(f,i) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

Assignment 6.2 (Exercise 3.8.7)

Prove that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \text{ whenever } a > 1.$$

Proof Let $z = \mathrm{e}^{i\theta}$ and let C be the unit circle. Then $\mathrm{d}z = iz\mathrm{d}\theta$ and $\cos\theta = (z+z^{-1})/2$, so

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} = \int_C \frac{\mathrm{d}z}{iz(a+(z+z^{-1})/2)^2} = \frac{4}{i} \int_C \frac{z\mathrm{d}z}{(z^2+2az+1)^2}.$$

The zeros of $z^2+2az+1$ are $-a \pm \sqrt{a^2-1}$. Since $a > 1$, only $z_0 = -a + \sqrt{a^2-1}$ lies inside C, and it is a pole of order 2. By the residue formula, we have

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} &= \frac{4}{i} \cdot 2\pi i \operatorname{Res}\left(\frac{z}{(z^2+2az+1)^2}, z_0 \right). = 8\pi \lim_{z\to z_0} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{(z+a+\sqrt{a^2-1})^2}. \\ &= 8\pi \lim_{z\to z_0} \frac{(z+a+\sqrt{a^2-1})^2 - 2z(z+a+\sqrt{a^2-1})}{(z+a+\sqrt{a^2-1})^4} = \frac{2\pi a}{(a^2-1)^{3/2}}, \text{ whenever } a > 1.\end{aligned}$$

Assignment 6.3 (Exercise 3.8.8)

Prove that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

if $a > |b|$ *and* $a, b \in \mathbb{R}$.

Proof Since replacing θ by $\theta+\pi$ changes b to $-b$, we may assume $b>0$ WLOG. Let $z=\mathrm{e}^{i\theta}$ and let C be the unit circle. Then $\mathrm{d}z=iz\mathrm{d}\theta$ and $\cos\theta=(z+z^{-1})/2$, so

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\frac{2}{bi}\int_C\frac{\mathrm{d}z}{z^2+2az/b+1}.$$

The zeros of $z^2+2az/b+1$ are $-a/b\pm\sqrt{(a/b)^2-1}$. Since $a>|b|$, only $z_0=-a/b+\sqrt{(a/b)^2-1}$ lies inside C and it is a simple pole. By the residue formula, we have

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\frac{2}{bi}\cdot 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2+2az/b+1},z_0\right).=\frac{4\pi}{b}\cdot\frac{1}{2\sqrt{(a/b)^2-1}}=\frac{2\pi}{\sqrt{a^2-b^2}}.$$

And the equality holds for all $a>|b|$, where $a,b\in\mathbb{R}$.

Assignment 6.4 (Exercise 3.8.14)

Prove that all entire functions that are also injective take the form $f(z)=az+b$ with $a,b\in\mathbb{C}$, and $a\neq 0$.

[Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.]

♠

Proof Assume that f is an injective entire function. Then f is non-constant, hence it is unbounded by Liouville's theorem. Let $g(z)=f(1/z)$. Then $z=0$ is either a pole or an essential singularity of g. We first show that 0 cannot be an essential singularity of g. If it were, then by Casorati-Weierstrass theorem, the set $g(0<|z|<1)=f(|w|>1)$ would be dense in $\mathbb{C}$. But we know that $f(|w|<1/2)$ is a nonempty open set by the open mapping theorem. Since f is injective, the sets $f(|w|>1)$ and $f(|w|<1/2)$ are disjoint. Thus $f(|w|>1)$ cannot be dense in $\mathbb{C}$, which leads to a contradiction. Therefore 0 is a pole of g, then we can write $g(z)=\sum_{n=-m}^{+\infty}a_nz^n$ for some $m\in\mathbb{N}^+$ and $a_{-m}\neq 0$, so $f(z)=\sum_{n=-m}^{+\infty}a_nz^{-n}=\sum_{n=-\infty}^{m}a_{-n}z^n$. Since f is entire, we can obtain $a_1=a_2=\cdots=0$, which means that f is a non-constant polynomial, then f has a zero z_0. Since f is injective, z_0 is the unique zero of f. So we can obtain $f(z)=a_{-m}(z-z_0)^m$. But if $m\geqslant 2$, we have $f(z_0+1)=a_{-m}=f(z_0+\mathrm{e}^{2\pi i/m})$ and $1\neq\mathrm{e}^{2\pi i/m}$, then f is not injective, which leads to a contradiction. Therefore we have $m=1$ and $f(z)=az+b$ with $a,b\in\mathbb{C}$ and $a\neq 0$. So we are done.

Assignment 6.5 (Exercise 3.8.15)

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R}|f(z)|\leqslant AR^k+B$$

for all $R>0$, and for some integer $k\geqslant 0$ and some constants $A,B>0$, then f is a polynomial of degree $\leqslant k$.

(b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta<\arg z<\varphi$ as $|z|\to 1$, then $f=0$.

(c) Let $w_1,\cdots,w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j, $1\leqslant j\leqslant n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points w_j, $1\leqslant j\leqslant n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

Proof (a) By the Cauchy inequalities, for every $n \geqslant 0$ and every $R > 0$ we have

$$|f^{(n)}(0)| \leqslant \frac{n!(AR^k + B)}{R^n}.$$

For $n > k$, letting $R \to +\infty$ gives $f^{(n)}(0) = 0$. Hence f is a polynomial of degree $\leqslant k$.
(b) we can choose $N \in \mathbb{N}$ so large that $N(\varphi - \theta)/2 > 2\pi$. Define $g(z) := \prod_{k=0}^{N} f(\mathrm{e}^{ik(\varphi-\theta)/2} z)$ and let $S = \{\theta < \arg z < \varphi\}$, then $\bigcup_{k=0}^{N} \mathrm{e}^{-ik(\varphi-\theta)/2} S \supset \mathbb{D}$. Let M be a bound for $|f|$ on $\mathbb{D}$. For any given $\varepsilon > 0$, there exists $0 < r < 1$ such that $|f(z)| < \varepsilon$ whenever $r < |z| < 1$ and $\theta < \arg z < \varphi$ because f converges uniformly to zero in S as $|z| \to 1$. Therefore, for every z with $r < |z| < 1$, at least one factor in the product defining $g(z)$ is $< \varepsilon$, so $|g(z)| < M^N \varepsilon$. By the maximum modulus principle, the same estimate holds on $|z| \leqslant r$ as well. Since $\varepsilon > 0$ is arbitrary, we get $g \equiv 0$ on $\mathbb{D}$. If f is not identically zero, then its zeros on $\mathbb{D}$ would be isolated, so the zero set of f is at most countable. Thus the zero set of g on $\mathbb{D}$, being a finite union of rotated zero sets of f, is at most countable as well and could not be all of $\mathbb{D}$, which leads to a contradiction. Therefore $f \equiv 0$.
(c) Consider the polynomial $p(z) = (z - w_1) \cdots (z - w_n)$. Since $|w_j| = 1, 1 \leqslant j \leqslant n$, we have $|p(0)| = |w_1 \cdots w_n| = 1$. Since p is non-constant, by the maximum modulus principle, there exists a point z on the unit circle such that $|p(z)| = \prod_{j=1}^{n} |z - w_j| \geqslant 1$. So this proves the first claim.

Then note that $|p(z)|$ is continuous on the unit circle with $|p(w_j)| = 0, 1 \leqslant j \leqslant n$ and $|p(z)| \geqslant 1$. Hence by the intermediate value theorem, there exists a point w on the unit circle such that $|p(w)| = \prod_{j=1}^{n} |w - w_j| = 1$. So this proves the second claim.
(d) Consider the function $g(z) = \mathrm{e}^{f(z)}$. Then g is entire and $|g(z)| = \mathrm{e}^{\operatorname{Re} f(z)}$. Since $\operatorname{Re} f$ is bounded, g is bounded. By Liouville's theorem, g is constant. Then we can obtain $g'(z) = f'(z)\mathrm{e}^{f(z)} = 0$, hence $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is constant.

Assignment 6.6 (Exercise 3.8.18)

Give another proof of the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

using homotopy of curves.
[Hint: Deform the circle C to a small circle centered at z, and note that the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded.]

Proof Let D be the disc bounded by C, for any fixed $z \in D$, we can choose $\varepsilon > 0$ so small that the positively oriented circle $C_\varepsilon := \{\zeta \in \mathbb{C} : |\zeta - z| = \varepsilon\}$ lies inside D. Choose a point $a \in C$ and let $b := z + \varepsilon(a - z)/|a - z| \in C_\varepsilon$. Now let $\gamma : [0, 1] \to \Omega$ be the straight line segment connecting a, b with $\gamma(0) = a$ and $\gamma(1) = b$, where Ω is the domain of f containing $\overline{D}$. And let $\alpha, \beta : [0, 1] \to \Omega$ be parametrizations of these two circles such that $\alpha(0) = \alpha(1) = a$ and $\beta(0) = \beta(1) = b$. Define two closed paths based at a by $\Gamma_1 := \alpha$ and $\Gamma_2 := \gamma * \beta * \gamma^{-1}$. It is clear that Γ_1 and Γ_2 are homotopic in the open set $\Omega \setminus \{z\}$. So for $h(\zeta) := f(\zeta)/(\zeta - z)$, which is holomorphic on $\Omega \setminus \{z\}$, by Theorem 5.1

of the textbook, we have

$$\int_{\Gamma_1} h(\zeta)\mathrm{d}\zeta = \int_{\Gamma_2} h(\zeta)\mathrm{d}\zeta = \int_\gamma h(\zeta)\mathrm{d}\zeta + \int_\beta h(\zeta)\mathrm{d}\zeta + \int_{\gamma^{-1}} h(\zeta)\mathrm{d}\zeta = \int_\beta h(\zeta)\mathrm{d}\zeta.$$

Therefore we can obtain

$$\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

Now we define $g(\zeta) := (f(\zeta) - f(z))/(\zeta - z)$. Then g extends holomorphically to $\zeta = z$ since f is holomorphic, so g is bounded near z. Hence

$$\left|\int_{C_\varepsilon} g(\zeta)\mathrm{d}\zeta\right| \leqslant 2\pi\varepsilon \sup_{|\zeta - z| = \varepsilon} |g(\zeta)| \to 0\,(\varepsilon \to 0).$$

On the other hand, we have

$$\int_{C_\varepsilon} \frac{f(z)}{(\zeta - z)}\mathrm{d}\zeta = f(z)\int_{C_\varepsilon} \frac{\mathrm{d}\zeta}{\zeta - z} = 2\pi i f(z).$$

Therefore, letting $\varepsilon \to 0$, we can obtain

$$\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = 2\pi i f(z) \Rightarrow f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

Assignment 6.7 (Exercise 3.8.20)

This exercise shows how the mean square convergence dominates the uniform convergence of analytic functions. If U is an open subset of $\mathbb{C}$ we use the notation

$$\|f\|_{L^2(U)} = \left(\int_U |f(z)|^2\mathrm{d}x\mathrm{d}y\right)^{1/2}$$

for the mean square norm, and

$$\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z)|$$

for the sup norm.

(a) If f is holomorphic in a neighborhood of the disc $D_r(z_0)$, show that for any $0 < s < r$ there exists a constant $C > 0$ (which depends on s and r) such that

$$\|f\|_{L^\infty(D_s(z_0))} \leqslant C\|f\|_{L^2(D_r(z_0))}.$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $\|\cdot\|_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly on every compact subset of U to a holomorphic function.

[Hint: Use the mean-value property.]

Proof (a) Fix $z \in D_s(z_0)$ and let $R = (r - s)/2$. Then $\overline{D_R(z)} \subset D_r(z_0)$. Since f is holomorphic on a neighborhood of $\overline{D_R(z)}$, for all $0 \leqslant \rho \leqslant R$, the mean-value property gives

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + \rho e^{i\theta})\mathrm{d}\theta \Rightarrow |f(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z + \rho e^{i\theta})|\mathrm{d}\theta.$$

By Cauchy-Schwarz inequality, we can obtain

$$|f(z)|^2 \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |f(z + \rho e^{i\theta})|\mathrm{d}\theta\right)^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z + \rho e^{i\theta})|^2\mathrm{d}\theta.$$

Now multiply both sides by 2ρ and integrate with respect to ρ from 0 to R, we have

$$|f(z)|^2R^2 = |f(z)|^2\int_0^R 2\rho\mathrm{d}\rho \leqslant \frac{1}{2\pi}\int_0^R\int_0^{2\pi}|f(z+\rho\mathrm{e}^{i\theta})|^2 2\rho\mathrm{d}\theta\mathrm{d}\rho = \frac{1}{\pi}\int_{D_R(z)}|f(\zeta)|^2\mathrm{d}x\mathrm{d}y.$$

Therefore we have

$$|f(z)|^2 \leqslant \frac{1}{\pi R^2}\int_{D_R(z)}|f(\zeta)|^2\mathrm{d}x\mathrm{d}y.$$

Since $R=(r-s)/2$ and $D_R(z)\subset D_r(z_0)$, it follows that

$$|f(z)|^2 \leqslant \frac{4}{\pi(r-s)^2}\int_{D_r(z_0)}|f(\zeta)|^2\mathrm{d}x\mathrm{d}y = \frac{4}{\pi(r-s)^2}\|f\|^2_{L^2(D_r(z_0))}.$$

Since this holds for every $z\in D_s(z_0)$, we conclude that

$$\|f\|_{L^\infty(D_s(z_0))} \leqslant \frac{2}{\sqrt{\pi}(r-s)}\|f\|_{L^2(D_r(z_0))}.$$

Thus the desired estimate holds with $C=2/(\sqrt{\pi}(r-s))$.

(b) Let $K\subset U$ be compact. Choose finitely many discs $D_{s_j}(z_j)$ such that $K\subset\bigcup_{j=1}^m D_{s_j}(z_j)\subset U$ and for each j, there exists $r_j>s_j$ with $\overline{D_{r_j}(z_j)}\subset U$. By (a), for every holomorphic h on a neighborhood of $\overline{D_{r_j}(z_j)}$ we have $\|h\|_{L^\infty(D_{s_j}(z_j))}\leqslant C_j\|h\|_{L^2(D_{r_j}(z_j))}\leqslant C_j\|h\|_{L^2(U)}$. Now apply this to $h=f_n-f_m$. Since $\{f_n\}$ is a Cauchy sequence in $L^2(U)$, it follows that $\{f_n\}$ is a Cauchy sequence in $L^\infty(D_{s_j}(z_j))$ for each j, and hence also in $L^\infty(K)$. Therefore $\{f_n\}$ converges uniformly on K. Since K is arbitrary, $\{f_n\}$ converges uniformly on every compact subset of U to some function f. Since $\{f_n\}$ is a sequence of holomorphic functions on U, we have f is also holomorphic on U.

Assignment 6.8 (Problem 5.6.3 of Book I)

The Dirichlet problem in a strip. Consider the equation $\Delta u=0$ in the horizontal strip

$$\{(x,y):0<y<1,\ -\infty<x<\infty\}$$

with boundary conditions $u(x,0)=f_0(x)$ and $u(x,1)=f_1(x)$, where f_0 and f_1 are both in the Schwartz space.

(a) Show (formally) that if u is a solution to this problem, then

$$\widehat{u}(\xi,y)=A(\xi)\mathrm{e}^{2\pi\xi y}+B(\xi)\mathrm{e}^{-2\pi\xi y}.$$

Express A and B in terms of $\widehat{f_0}$ and $\widehat{f_1}$, and show that

$$\widehat{u}(\xi,y)=\frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)}\widehat{f_0}(\xi)+\frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)}\widehat{f_1}(\xi).$$

(b) Prove as a result that

$$\int_{-\infty}^{\infty}|u(x,y)-f_0(x)|^2\mathrm{d}x\to 0 \text{ as } y\to 0,$$

and

$$\int_{-\infty}^{\infty}|u(x,y)-f_1(x)|^2\mathrm{d}x\to 0 \text{ as } y\to 1.$$

(c) If $\Phi(\xi)=(\sinh 2\pi\alpha\xi)/(\sinh 2\pi\xi)$, with $0\leqslant\alpha<1$, then Φ is the Fourier transform of φ where

$$\varphi(x)=\frac{\sin\pi\alpha}{2}\cdot\frac{1}{\cosh\pi x+\cos\pi\alpha}.$$

This can be shown, for instance, by using contour integration and the residue formula from complex analysis (see Book II, Chapter 3).

(d) Use this result to express u in terms of Poisson-like integrals involving f_0 and f_1 as follows:

$$u(x,y)=\frac{\sin\pi y}{2}\left(\int_{-\infty}^{\infty}\frac{f_0(x-t)}{\cosh\pi t-\cos\pi y}\mathrm{d}t+\int_{-\infty}^{\infty}\frac{f_1(x-t)}{\cosh\pi t+\cos\pi y}\mathrm{d}t\right).$$

(e) Finally, one can check that the function $u(x,y)$ defined by the above expression is harmonic in the strip, and converges uniformly to $f_0(x)$ as $y\to 0$, and to $f_1(x)$ as $y\to 1$. Moreover, one sees that $u(x,y)$ vanishes at infinity, that is, $\lim_{|x|\to\infty}u(x,y)=0$, uniformly in y.

In Exercise 12, we gave an example of a function that satisfies the heat equation in the upper half-plane, with boundary value 0, but which was not identically 0. We observed in this case that u was in fact not continuous up to the boundary.

In Problem 4 we exhibit examples illustrating non-uniqueness, but this time with continuity up to the boundary $t=0$. These examples satisfy a growth condition at infinity, namely $|u(x,t)|\leqslant Ce^{cx^{2+\varepsilon}}$, for any $\varepsilon>0$. Problems 5 and 6 show that under the more restrictive growth condition $|u(x,t)|\leqslant Ce^{cx^2}$, uniqueness does hold.

Proof We use the Fourier transform in the x-variable with the convention

$$\widehat{h}(\xi)=\int_{-\infty}^{\infty}h(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x.$$

(a) Taking Fourier transforms in x, the equation $u_{xx}+u_{yy}=0$ becomes $\partial_{yy}\widehat{u}(\xi,y)-4\pi^2\xi^2\widehat{u}(\xi,y)=0$. For each fixed ξ, this is an ODE in y, so formally $\widehat{u}(\xi,y)=A(\xi)\mathrm{e}^{2\pi\xi y}+B(\xi)\mathrm{e}^{-2\pi\xi y}$. The boundary conditions give $A(\xi)+B(\xi)=\widehat{f_0}(\xi)$, $A(\xi)\mathrm{e}^{2\pi\xi}+B(\xi)\mathrm{e}^{-2\pi\xi}=\widehat{f_1}(\xi)$. Solving this system, we have

$$A(\xi)=\frac{\widehat{f_1}(\xi)-\mathrm{e}^{-2\pi\xi}\widehat{f_0}(\xi)}{\mathrm{e}^{2\pi\xi}-\mathrm{e}^{-2\pi\xi}},\ B(\xi)=\frac{\mathrm{e}^{2\pi\xi}\widehat{f_0}(\xi)-\widehat{f_1}(\xi)}{\mathrm{e}^{2\pi\xi}-\mathrm{e}^{-2\pi\xi}}.$$

Substituting into the formula for $\widehat{u}$ and rewriting in terms of sinh, we get

$$\widehat{u}(\xi,y)=\frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)}\widehat{f_0}(\xi)+\frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)}\widehat{f_1}(\xi).$$

(b) By Plancherel's theorem, we have

$$\|u(\cdot,y)-f_0\|^2_{L^2(\mathbb{R})}=\int_{-\infty}^{\infty}\left|\left(\frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)}-1\right)\widehat{f_0}(\xi)+\frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)}\widehat{f_1}(\xi)\right|^2\mathrm{d}\xi.$$

For each fixed $\xi\neq 0$, the two coefficients tend to 0 as $y\to 0$. Moreover, for $0<y<1$ we have

$$0\leqslant\frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)}\leqslant 1,\ 0\leqslant\frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)}\leqslant 1$$

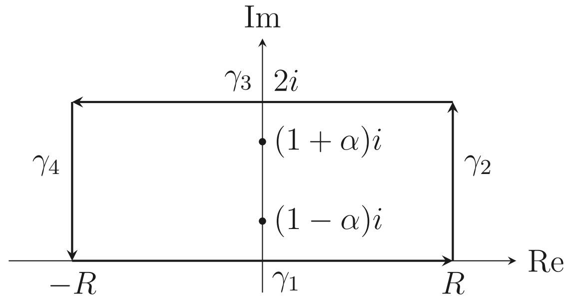
for $\xi\neq 0$, and the same is true at $\xi=0$ by continuity. So the integral is dominated by $2(|\widehat{f_0}(\xi)|^2+|\widehat{f_1}(\xi)|^2)$, which is integrable since f_0,f_1 are in the Schwartz space. So by DCT, $\|u(\cdot,y)-f_0\|^2_{L^2(\mathbb{R})}\to 0$ as $y\to 0$. The proof of $\|u(\cdot,y)-f_1\|^2_{L^2(\mathbb{R})}\to 0$ as $y\to 1$ is exactly the same.

(c) Let $f(z)=\mathrm{e}^{-2\pi iz\xi}\sin\pi\alpha/(2\cosh\pi z+2\cos\pi\alpha)$, where $0\leqslant\alpha<1$. Then we note that

$$\cosh\pi z+\cos\pi\alpha=0\iff\frac{\mathrm{e}^{\pi z}+\mathrm{e}^{-\pi z}}{2}+\cos\pi\alpha=0\iff\mathrm{e}^{2\pi z}+(2\cos\pi\alpha)\mathrm{e}^{\pi z}+1=0$$

$$\iff\mathrm{e}^{\pi z}=-\cos\pi\alpha\pm i\sin\pi\alpha=-\mathrm{e}^{\mp i\pi\alpha}\iff\mathrm{e}^{\pi(z\pm i\alpha)}=-1\iff z=(2n+1\mp\alpha)i\text{ for }n\in\mathbb{Z}.$$

When $0 \leqslant \alpha < 1$, we have$(\cosh \pi z + \cos \pi\alpha)'\big|_{z=(2n+1\mp\alpha)i} = \pi \sinh(\pi(2n+1\mp\alpha)i) \neq 0$. Thus, the poles of $f(z)$ are all simple. Now consider the integral of $f(z)$ over the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as shown below.



Observe that $(1 \pm \alpha)i$ are the only two poles of $f(z)$ in the region enclosed by the contour γ. The corresponding residues are given by

$$\operatorname{Res}(f,(1-\alpha)i) = \lim_{z\to(1-\alpha)i} (z-(1-\alpha)i)f(z)$$

$$= \frac{\mathrm{e}^{-2\pi i(1-\alpha)i\xi}\sin\pi\alpha}{2\pi\sinh(\pi(1-\alpha)i)} = \frac{\mathrm{e}^{2\pi(1-\alpha)\xi}\sin\pi\alpha}{2\pi i\sin(\pi\alpha)} = \frac{\mathrm{e}^{2\pi(1-\alpha)\xi}}{2\pi i},$$

$$\operatorname{Res}(f,(1+\alpha)i) = \lim_{z\to(1+\alpha)i} (z-(1+\alpha)i)f(z)$$

$$= \frac{\mathrm{e}^{-2\pi i(1+\alpha)i\xi}\sin\pi\alpha}{2\pi\sinh(\pi(1+\alpha)i)} = -\frac{\mathrm{e}^{2\pi(1+\alpha)\xi}\sin\pi\alpha}{2\pi i\sin(\pi\alpha)} = -\frac{\mathrm{e}^{2\pi(1+\alpha)\xi}}{2\pi i}.$$

On γ_2, write $z = R + iy$ with $0 \leqslant y \leqslant 2$ and $R > 0$, then we have $|\mathrm{e}^{-2\pi i z\xi}| = \mathrm{e}^{2\pi y\xi} \leqslant \mathrm{e}^{4\pi|\xi|}$ and

$$|\cosh \pi z| = \left|\frac{\mathrm{e}^{\pi z}+\mathrm{e}^{-\pi z}}{2}\right| \geqslant \frac{1}{2}\big||\mathrm{e}^{\pi z}| - |\mathrm{e}^{-\pi z}|\big| = \frac{1}{2}\left(\mathrm{e}^{\pi R} - \mathrm{e}^{-\pi R}\right) \to +\infty \, (R \to +\infty).$$

which show that the integral over γ_2 goes to 0 as $R \to +\infty$. A similar argument applies to γ_4. Finally, we see that if I denotes the integral we wish to calculate, then the integral of f over γ_3 tends to

$$\lim_{R\to+\infty} \int_R^{-R} \mathrm{e}^{-2\pi i(x+2i)\xi} \frac{\sin\pi\alpha}{2\cosh\pi(x+2i) + 2\cos\pi\alpha} \mathrm{d}x = -\mathrm{e}^{4\pi\xi} I.$$

Letting $R \to +\infty$, the residue formula gives $I - \mathrm{e}^{4\pi\xi} I = \mathrm{e}^{2\pi(1-\alpha)\xi} - \mathrm{e}^{2\pi(1+\alpha)\xi} = -2\mathrm{e}^{2\pi\xi}\sinh 2\pi\alpha\xi$, and since $1 - \mathrm{e}^{4\pi\xi} = -\mathrm{e}^{2\pi\xi}(\mathrm{e}^{2\pi\xi} - \mathrm{e}^{-2\pi\xi}) = -2\mathrm{e}^{2\pi\xi}\sinh 2\pi\xi$, we find that

$$I = \widehat{\varphi}(\xi) = \frac{-2\mathrm{e}^{2\pi\xi}\sinh 2\pi\alpha\xi}{-2\mathrm{e}^{2\pi\xi}\sinh 2\pi\xi} = \frac{\sinh 2\pi\alpha\xi}{\sinh 2\pi\xi} = \Phi(\xi).$$

(d) Applying part (c) with $\alpha = 1 - y$ and $\alpha = y$, we obtain

$$\frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)} = \widehat{\varphi_{1-y}}(\xi), \ \frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)} = \widehat{\varphi_y}(\xi),$$

where

$$\varphi_{1-y}(t) = \frac{\sin\pi(1-y)}{2(\cosh\pi t + \cos\pi(1-y))} = \frac{\sin\pi y}{2(\cosh\pi t - \cos\pi y)}, \ \varphi_y(t) = \frac{\sin\pi y}{2(\cosh\pi t + \cos\pi y)}.$$

Hence, by the convolution theorem and part (a), $u(\cdot, y) = \varphi_{1-y} * f_0 + \varphi_y * f_1$. That is to say

$$u(x,y) = \frac{\sin\pi y}{2}\left(\int_{-\infty}^{\infty} \frac{f_0(x-t)}{\cosh\pi t - \cos\pi y}\mathrm{d}t + \int_{-\infty}^{\infty} \frac{f_1(x-t)}{\cosh\pi t + \cos\pi y}\mathrm{d}t\right).$$

(e) The formula in part (d) has Fourier transform exactly equal to the expression found in part (a), so it satifies the same transformed boundary-value problem. Therefore u is harmonic in the strip.

Now φ_{1-y} is an approximate identity as $y \to 0$, while $\varphi_y \to 0$ in $L^1(\mathbb{R})$ as $y \to 0$. Since f_0, f_1 are Schwartz functions, hence bounded and uniformly continuous, it follows that $\varphi_{1-y} * f_0 \to f_0$ uniformly and $\varphi_y * f_1 \to 0$ uniformly. Thus $u(x, y) \to f_0(x)$ uniformly as $y \to 0$. Similarly, $u(x, y) \to f_1(x)$ uniformly as $y \to 1$.

Finally, since f_0 and f_1 are Schwartz functions and the kernels φ_{1-y}, φ_y have uniformly bounded L^1 norms for $0 < y < 1$, the convolutions $\varphi_{1-y} * f_0$ and $\varphi_y * f_1$ tend to 0 as $|x| \to \infty$, uniformly in y. Hence $\lim_{|x| \to \infty} u(x, y) = 0$ uniformly in y.

Chapter 7 Homework 7

Assignment 7.1 (Exercise 3.8.22)

Show that there is no holomorphic function f in the unit disc $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$.

Proof Assume such an f exists. Then Let $g(z) = zf(z)$, we have g is holomorphic in $\mathbb{D}$ and extends continuously to $\partial\mathbb{D}$. For $z \in \partial\mathbb{D}$, we have $f(z) = 1/z$, hence $g(z) = 1$ on $\partial\mathbb{D}$. So $g-1$ is holomorphic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and vanishes on $\partial\mathbb{D}$. By the maximum modulus principle, $g - 1 \equiv 0$ in $\mathbb{D}$, so $g \equiv 1$. But $g(0) = 0 \cdot f(0) = 0$, which is impossible. This contradiction shows that no such f exists.

Assignment 7.2 (Problem 3.9.3)

If $f(z)$ is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{z-z_0} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$. There is a generalization of this expansion that holds even if z_0 is an essential singularity. This is a special case of the ***Laurent series expansion****, which is valid in an even more general setting.*
Let f be holomorphic in a region containing the annulus $\{z : r_1 \leqslant |z - z_0| \leqslant r_2\}$ where $0 < r_1 < r_2$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the series converges absolutely in the interior of the annulus. To prove this, it suffices to write

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

when $r_1 < |z - z_0| < r_2$, and argue as in the proof of Theorem 4.4, Chapter 2. Here C_{r_1} and C_{r_2} are the circles bounding the annulus.

Proof Fix z with $r_1 < |z - z_0| < r_2$. For $\zeta \in C_{r_2}$ we have $|z - z_0| < |\zeta - z_0|$, so

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}.$$

And the series converges absolutely and uniformly on C_{r_2}. Hence

$$\frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\mathrm{d}\zeta \right)(z - z_0)^n.$$

For $\zeta \in C_{r_1}$ we have $|\zeta - z_0| < |z - z_0|$, so

$$\frac{1}{\zeta - z} = -\frac{1}{(z - z_0) - (\zeta - z_0)} = -\sum_{m=0}^{\infty} \frac{(\zeta - z_0)^m}{(z - z_0)^{m+1}}.$$

Again the series converges absolutely and uniformly on C_{r_1}. Hence

$$-\frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta=\sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_{r_1}}f(\zeta)(\zeta-z_0)^m\mathrm{d}\zeta\right)(z-z_0)^{-m-1}.$$

Choose $\varepsilon>0$ sufficiently small such that the circle $C_\varepsilon:=\{\zeta:|\zeta-z|=\varepsilon\}$ lies entirely in the interior of the annulus, i.e., $\{\zeta:r_1<|\zeta-z_0|<r_2\}$. Let $D=\{\zeta:r_1\leqslant|\zeta-z_0|\leqslant r_2\}\setminus\{\zeta:|\zeta-z|<\varepsilon\}$. Then $f(\zeta)/(\zeta-z)$ is holomorphic on a neighborhood of D. Connect C_{r_1}, C_{r_2} and C_ε by three curves lying in the interior of the annulus such that D is divided into two simply connected regions. For each of the two regions, applying Corollary 5.3 of Chapter 3 to a neighborhood of it, we can obtain

$$\int_{\partial D}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta=0=\int_{C_{r_2}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta-\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta-\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta.$$

Therefore we can obtain

$$\int_{C_{r_2}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta-\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta=\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta=2\pi i f(z)$$

by Cauchy's integral formula. Thus we have

$$\begin{aligned}f(z)&=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta-\frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\\&=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{m=0}^{\infty}a_{-m-1}(z-z_0)^{-m-1}=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n\end{aligned}$$

where the coefficients are given by

$$a_n=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\mathrm{d}\zeta,\ a_{-m-1}=\frac{1}{2\pi i}\int_{C_{r_1}}f(\zeta)(\zeta-z_0)^m\mathrm{d}\zeta.$$

To prove absolute convergence in the interior of the annulus, choose numbers ρ_1,ρ_2 such that $r_1<\rho_1<|z-z_0|<\rho_2<r_2$. Let $M_2=\max_{\zeta\in C_{r_2}}|f(\zeta)|$, $M_1=\max_{\zeta\in C_{r_1}}|f(\zeta)|$. Since the length of C_{r_2} is $2\pi r_2$, we can obtain

$$|a_n|\leqslant\frac{1}{2\pi}\int_{C_{r_2}}\frac{|f(\zeta)|}{|\zeta-z_0|^{n+1}}|\mathrm{d}\zeta|\leqslant\frac{1}{2\pi}\cdot 2\pi r_2\cdot\frac{M_2}{r_2^{n+1}}=\frac{M_2}{r_2^n}.$$

$$|a_n(z-z_0)^n|\leqslant M_2\left(\frac{|z-z_0|}{r_2}\right)^n\leqslant M_2\left(\frac{\rho_2}{r_2}\right)^n.$$

Since $\rho_2/r_2<1$, $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ converges absolutely. Similarly, we have

$$|a_{-m-1}|\leqslant\frac{1}{2\pi}\int_{C_{r_1}}|f(\zeta)|\,|\zeta-z_0|^m\,|\mathrm{d}\zeta|\leqslant\frac{1}{2\pi}\cdot 2\pi r_1\cdot M_1r_1^m=M_1r_1^{m+1}.$$

$$|a_{-m-1}(z-z_0)^{-m-1}|\leqslant M_1\left(\frac{r_1}{|z-z_0|}\right)^{m+1}\leqslant M_1\left(\frac{r_1}{\rho_1}\right)^{m+1}.$$

Since $r_1/\rho_1<1$, $\sum_{m=0}^{\infty}a_{-m-1}(z-z_0)^{-m-1}$ also converges absolutely. Combining the two parts, we conclude that the series $\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ converges absolutely in the interior of the annulus.

Assignment 7.3 (Exercise 4.4.1)

Suppose f is continuous and of moderate decrease, and $\widehat{f}(\xi)=0$ for all $\xi\in\mathbb{R}$. Show that $f=0$ by completing the following outline:

(a) For each fixed real number t consider the two functions

$$A(z)=\int_{-\infty}^{t} f(x)\mathrm{e}^{-2\pi iz(x-t)}\mathrm{d}x \text{ and } B(z)=-\int_{t}^{\infty} f(x)\mathrm{e}^{-2\pi iz(x-t)}\mathrm{d}x.$$

Show that $A(\xi)=B(\xi)$ for all $\xi\in\mathbb{R}$.

(b) Prove that the function F equal to A in the closed upper half-plane, and B in the lower half-plane, is entire and bounded, thus constant. In fact, show that $F=0$.

(c) Deduce that

$$\int_{-\infty}^{t} f(x)\mathrm{d}x=0,$$

for all t, and conclude that $f=0$.

Proof (a) For $\xi\in\mathbb{R}$ we have

$$A(\xi)-B(\xi)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i\xi(x-t)}\mathrm{d}x=\mathrm{e}^{2\pi i\xi t}\widehat{f}(\xi)=0.$$

Hence $A(\xi)=B(\xi)$ for all $\xi\in\mathbb{R}$.

(b) Since A and B are holomorphic in the upper and lower half-planes respectively and agree on the real axis, the function $F(z)$ is entire by the symmetry principle. Since f is of moderate decrease, there is a constant $C>0$ such that $|f(x)|\leqslant C/(1+x^2)$ for all $x\in\mathbb{R}$. If $\operatorname{Im} z\geqslant 0$, we have

$$|A(z)|\leqslant\int_{-\infty}^{t}|f(x)|\mathrm{e}^{2\pi\operatorname{Im}(z)(x-t)}\mathrm{d}x\leqslant\int_{-\infty}^{t}\frac{C}{1+x^2}\mathrm{d}x\leqslant\pi C.$$

Similarly, if $\operatorname{Im} z<0$, we have

$$|B(z)|\leqslant\int_{t}^{\infty}|f(x)|\mathrm{e}^{2\pi\operatorname{Im}(z)(x-t)}\mathrm{d}x\leqslant\int_{t}^{\infty}\frac{C}{1+x^2}\mathrm{d}x\leqslant\pi C.$$

Thus F is a bounded entire function, hence constant by Liouville's theorem. Now for $s>0$, we have

$$A(is)=\int_{-\infty}^{t} f(x)\mathrm{e}^{2\pi s(x-t)}\mathrm{d}x\to 0\,(s\to+\infty)$$

by DCT. Since F is constant, it follows that $F\equiv 0$.

(c) Since $F(0)=0$, we have

$$0=A(0)=\int_{-\infty}^{t} f(x)\mathrm{d}x\triangleq H(t)$$

for all $t\in\mathbb{R}$. Then $H(t)\equiv 0$. Since f is continuous, $H'(t)=f(t)=0$ for all $t\in\mathbb{R}$, so $f=0$.

Assignment 7.4 (Exercise 4.4.2)

If $f\in\mathfrak{F}_a$ with $a>0$, then for any positive integer n one has $f^{(n)}\in\mathfrak{F}_b$ whenever $0<b<a$. [Hint: Modify the solution to Exercise 8 in Chapter 2.]

Proof Fix b with $0<b<a$, and let $r=a-b>0$. It is clear that $f^{(n)}$ is holomorphic in the strip S_b. Thus only the condition (ii) needs to be checked. Let $z=x+iy$ with $x,y\in\mathbb{R}$ and $|y|<b$. Then $\overline{B(z,r)}$ lies in the strip S_a. By the Cauchy inequalities, we have

$$|f^{(n)}(z)|\leqslant\frac{n!}{r^n}\sup_{|\zeta-z|=r}|f(\zeta)|.$$

Since $f\in\mathfrak{F}_a$, there is a constant $A>0$ such that $|f(x+iy)|\leqslant A/(1+x^2)$ for all $x\in\mathbb{R}$ and $|y|<a$.

Therefore

$$|f^{(n)}(x+iy)| \leqslant \frac{n!A}{r^n} \sup_{\theta\in[0,2\pi]} \frac{1}{1+(x+r\cos\theta)^2} = \frac{n!A}{r^n} \cdot \frac{1}{1+(|x|-r)^2}.$$

Now, since we have

$$\frac{1+x^2}{1+(|x|-r)^2} \to 1\,(|x|\to\infty),$$

so there exists a constant $C>0$ such that $1+(|x|-r)^2 \geqslant C(1+x^2)$ for all $x\in\mathbb{R}$. Hence

$$|f^{(n)}(x+iy)| \leqslant \frac{n!A}{Cr^n}\cdot\frac{1}{1+x^2}$$

for all $x\in\mathbb{R}$ and $|y|<b$. Thus for any positive integer n, we have $f^{(n)}\in\mathfrak{F}_b$ whenever $0<b<a$.

Assignment 7.5 (Exercise 4.4.3)

Show, by contour integration, that if $a>0$ and $\xi\in\mathbb{R}$ then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi a|\xi|}\mathrm{e}^{2\pi i\xi x}\mathrm{d}\xi = \frac{1}{\pi}\cdot\frac{a}{a^2+x^2}.$$

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Proof Assume that $\xi\leqslant 0$ WLOG. The case $\xi=0$ is immediate. Now let $\xi<0$, define

$$f(z)=\frac{a}{a^2+z^2}\mathrm{e}^{-2\pi iz\xi}.$$

and consider the standard upper semicircular contour $\gamma=\gamma_1\cup\gamma_2$ of radius $R>a>0$. The only pole in the upper half-plane is at $z=ai$, and it is simple. Thus

$$\operatorname{Res}(f,ai)=\lim_{z\to ai}\frac{a}{z+ai}\mathrm{e}^{-2\pi iz\xi}=\frac{1}{2i}\mathrm{e}^{2\pi a\xi}.$$

Hence by the residue formula, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i\operatorname{Res}(f,ai)=\pi\mathrm{e}^{2\pi a\xi}=\pi\mathrm{e}^{-2\pi a|\xi|}.$$

On γ_2, write $z=Re^{i\theta}$, $0\leqslant\theta\leqslant\pi$. Since $\xi<0$, $|\mathrm{e}^{-2\pi iz\xi}|=\mathrm{e}^{2\pi R\xi\sin\theta}\leqslant 1$, so for $R>a$, we have

$$\left|\int_{\gamma_2}f(z)\mathrm{d}z\right|\leqslant\int_0^\pi\frac{aR}{R^2-a^2}\mathrm{d}\theta=\frac{\pi aR}{R^2-a^2}\to 0\,(R\to+\infty).$$

Therefore we can obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x=\frac{1}{\pi}\cdot\pi\mathrm{e}^{-2\pi a|\xi|}=\mathrm{e}^{-2\pi a|\xi|}.$$

For the second equality, by Lemma 2.3 of the textbook, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi a|\xi|}\mathrm{e}^{2\pi i\xi x}\mathrm{d}\xi\\ =&\int_{-\infty}^{0}\mathrm{e}^{2\pi(a+ix)\xi}\mathrm{d}\xi+\int_0^{\infty}\mathrm{e}^{-2\pi(a-ix)\xi}\mathrm{d}\xi=\int_0^{\infty}\mathrm{e}^{-2\pi(a+ix)\xi}\mathrm{d}\xi+\int_0^{\infty}\mathrm{e}^{-2\pi(a-ix)\xi}\mathrm{d}\xi\\ =&\frac{1}{2\pi(a+ix)}+\frac{1}{2\pi(a-ix)}=\frac{1}{\pi}\cdot\frac{a}{a^2+x^2}.\end{aligned}$$

Therefore, we have proved the two required equalities, as desired.

Assignment 7.6 (Exercise 4.4.4)

Suppose Q is a polynomial of degree $\geqslant 2$ with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{Q(x)}\mathrm{d}x, \xi \in \mathbb{R}$$

in terms of the roots of Q. What happens when several roots coincide?
[Hint: Consider separately the cases $\xi < 0$, $\xi = 0$, and $\xi > 0$. Use residues.]

Proof Let U be the set of roots of Q in the upper half-plane and L the set of roots of Q in the lower half-plane. Assume first that $\xi \leqslant 0$. Let

$$f(z) = \frac{\mathrm{e}^{-2\pi i z\xi}}{Q(z)}$$

and consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > R_0 > 0$, where R_0 is large enough that $|Q(z)| \geqslant C|z|^2$ for some constant $C > 0$ and all $|z| \geqslant R_0$ (since $\deg Q \geqslant 2$), and all the roots of $Q(z)$ are contained in $B(0, R_0)$. The poles in the upper half-plane are precisely the points in U, and they are all simple since the roots are distinct. Thus

$$\mathrm{Res}(f, \alpha) = \frac{\mathrm{e}^{-2\pi i\alpha\xi}}{Q'(\alpha)} \text{ for all } \alpha \in U.$$

Hence by the residue formula, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{\alpha\in U} \mathrm{Res}(f, \alpha) = 2\pi i \sum_{\alpha\in U} \frac{\mathrm{e}^{-2\pi i\alpha\xi}}{Q'(\alpha)}.$$

On γ_2, write $z = R\mathrm{e}^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Since $\xi \leqslant 0$, $|\mathrm{e}^{-2\pi i z\xi}| = \mathrm{e}^{2\pi R\xi \sin\theta} \leqslant 1$, so for $R \geqslant R_0$, we have

$$\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| \leqslant \int_0^\pi \frac{R}{CR^2}\mathrm{d}\theta = \frac{\pi}{CR} \to 0\,(R \to +\infty).$$

Therefore we can obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{Q(x)}\mathrm{d}x = \lim_{R\to+\infty} \int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{\alpha\in U} \mathrm{Res}(f, \alpha) = 2\pi i \sum_{\alpha\in U} \frac{\mathrm{e}^{-2\pi i\alpha\xi}}{Q'(\alpha)}, \ \xi \leqslant 0.$$

If $\xi \geqslant 0$, consider the polynomial $Q(-x)$ and substitute $-x$ for x. Then we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi i x\xi}}{Q(x)}\mathrm{d}x = -2\pi i \sum_{\beta\in L} \frac{\mathrm{e}^{-2\pi i\beta\xi}}{Q'(\beta)}, \ \xi \geqslant 0.$$

If several roots coincide, the same contour argument still works, but the residues at poles of higher order are computed by higher derivatives.

Assignment 7.7 (Exercise 4.4.5)

More generally, let $R(x) = P(x)/Q(x)$ be a rational function with (degree Q) $\geqslant$ (degree P)+2 and $Q(x) \neq 0$ on the real axis.
(a) Prove that if $\alpha_1, \cdots, \alpha_k$ are the roots of Q in the upper half-plane, then there exists polynomials $P_j(\xi)$ of degree less than the multiplicity of α_j so that

$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi i x\xi}\mathrm{d}x = \sum_{j=1}^{k} P_j(\xi)\mathrm{e}^{-2\pi i\alpha_j\xi}, \text{ when } \xi < 0.$$

(b) In particular, if $Q(z)$ has no zeros in the upper half-plane, then
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = 0 \text{ for } \xi < 0.$$
(c) Show that similar results hold in the case $\xi > 0$.
(d) Show that
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = O(\mathrm{e}^{-a|\xi|}), \xi \in \mathbb{R}$$
as $|\xi| \to \infty$ for some $a > 0$. Determine the best possible a's in terms of the roots of R.
[Hint: For part (a), use residues. The powers of ξ appear when one differentiates the function $f(z) = R(z)\mathrm{e}^{-2\pi iz\xi}$ (as in the formula of Theorem 1.4 in the previous chapter). For part (c) argue in the lower half-plane.]

Proof (a) When $\xi < 0$, let $f(z) = R(z)\mathrm{e}^{-2\pi iz\xi}$ and consider the standard upper semi-circular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $\rho > R_0 > 0$, where R_0 is large enough that $|R(z)| \leqslant C/|z|^2$ for some constant $C > 0$ and all $|z| \geqslant R_0$ (since $\deg Q \geqslant \deg P + 2$), and all the roots of $Q(z)$ in the upper half-plane are contained in $B(0, R_0)$. On γ_2, write $z = \rho \mathrm{e}^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Since $\xi < 0$, $|\mathrm{e}^{-2\pi iz\xi}| = \mathrm{e}^{2\pi\rho\xi\sin\theta} \leqslant 1$, so for $\rho \geqslant R_0$, we can obtain
$$\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| \leqslant \int_0^{\pi} \frac{C\rho}{\rho^2}\mathrm{d}\theta = \frac{C\pi}{\rho} \to 0\,(\rho \to +\infty).$$
Hence by the residue formula, we have
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = \lim_{\rho\to+\infty}\int_{\gamma} f(z)\mathrm{d}z = 2\pi i\sum_{j=1}^{k}\mathrm{Res}(f, \alpha_j).$$
If α_j has multiplicity m_j, write
$$R(z) = \sum_{l=1}^{m_j}\frac{c_{j,l}}{(z-\alpha_j)^l} + h_j(z)$$
where h_j is holomorphic near α_j. Therefore $\mathrm{Res}(f, \alpha_j)$ is of the form $Q_j(\xi)\mathrm{e}^{-2\pi i\alpha_j\xi}$, where Q_j is a polynomial of degree $\leqslant m_j - 1$. Let $P_j(\xi) = 2\pi iQ_j(\xi)$, $\deg P_j < m_j$, then we have
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = \sum_{j=1}^{k} P_j(\xi)\mathrm{e}^{-2\pi i\alpha_j\xi},\ \xi < 0.$$
(b) If $Q(z)$ has no zeros in the upper half-plane, the sum in (a) is empty, so the integral is 0 for $\xi < 0$.
(c) For $\xi > 0$, consider the standard lower semicircular contour $\widetilde{\gamma} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$ of radius $\rho > R_0 > 0$. The same argument gives
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = \sum_{\beta\in L}\widetilde{P}_\beta(\xi)\mathrm{e}^{-2\pi i\beta\xi},\ \xi > 0$$
where L is the set of roots of Q in the lower half-plane and $\deg \widetilde{P}_\beta$ is less than the multiplicity of β for every $\beta \in L$. Hence the similar results hold in the case $\xi > 0$.
(d) Let $\delta_+ = \min\{\mathrm{Im}\,\alpha : \alpha \text{ is a pole of } R,\ \mathrm{Im}\,\alpha > 0\}, \delta_- = \min\{-\mathrm{Im}\,\beta : \beta \text{ is a pole of } R,\ \mathrm{Im}\,\beta < 0\}$. For $\xi < 0$, (a) shows that the integral is a finite sum of terms $P_j(\xi)\mathrm{e}^{-2\pi i\alpha_j\xi}$, whose absolute values

are bounded by $C(1+|\xi|^N)\mathrm{e}^{-2\pi\delta_+|\xi|}$. Hence
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = O(\mathrm{e}^{-a|\xi|})\ (\xi\to-\infty)$$
for every $0 < a < 2\pi\delta_+$. Similarly, for $\xi > 0$ we obtain the same estimate for every $0 < a < 2\pi\delta_-$. Therefore, for $\xi\in\mathbb{R}$, as $|\xi|\to\infty$, we have
$$\int_{-\infty}^{\infty} R(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = O(\mathrm{e}^{-a|\xi|})$$
for every $0 < a < 2\pi\min(\delta_+,\delta_-)$. Since the poles nearest to the real axis determine the dominant exponential decay, the best possible a is $2\pi\min(\delta_+,\delta_-)^-$. Moreover, if all poles of R are simple, then this supremum is attained.

Chapter 8 Homework 8

Assignment 8.1 (Exercise 4.4.7)

The Poisson summation formula applied to specific examples often provides interesting identities.

(a) Let τ be fixed with $\operatorname{Im}(\tau) > 0$. Apply the Poisson summation formula to

$$f(z) = (\tau + z)^{-k},$$

where k is an integer $\geqslant 2$, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \mathrm{e}^{2\pi i m \tau}.$$

(b) Set $k = 2$ in the above formula to show that if $\operatorname{Im}(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can one conclude that the above formula holds true whenever τ is any complex number that is not an integer?

[Hint: For (a), use residues to prove that $\widehat{f}(\xi) = 0$, if $\xi < 0$, and

$$\widehat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} \mathrm{e}^{2\pi i \xi \tau}, \text{ when } \xi > 0.]$$

Proof (a) We compute the Fourier transform of f first. For $\xi \leqslant 0$, we have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} (\tau + x)^{-k} \mathrm{e}^{-2\pi i x \xi} \mathrm{d}x.$$

Consider the standard upper semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ of radius $R > |\tau| > 0$. Since $-\tau$ lies in the lower half-plane, the integrand is holomorphic in the upper half-plane. And on γ_2, we have

$$\left| \int_{\gamma_2} (\tau + z)^{-k} \mathrm{e}^{-2\pi i z \xi} \mathrm{d}z \right| \leqslant \int_0^{\pi} \frac{R \mathrm{e}^{2\pi R \xi \sin\theta}}{(R - |\tau|)^k} \mathrm{d}\theta \leqslant \frac{\pi R}{(R - |\tau|)^k} \to 0 \, (R \to +\infty)$$

since $\xi \leqslant 0$ and $k \geqslant 2$. Hence $\widehat{f}(\xi) = 0$ for $\xi \leqslant 0$. If $\xi > 0$, consider the standard lower semicircular contour $\widetilde{\gamma} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$ of radius $R > |\tau| > 0$ (anticlockwise). Then the only pole inside is at $z = -\tau$, and it has order k. The residue at $-\tau$ is

$$\operatorname{Res}\left((\tau+z)^{-k} \mathrm{e}^{-2\pi i z \xi}, -\tau\right) = \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \mathrm{e}^{-2\pi i z \xi} \bigg|_{z=-\tau} = \frac{(-2\pi i \xi)^{k-1}}{(k-1)!} \mathrm{e}^{2\pi i \xi \tau}.$$

Similarly, the integral over $\widetilde{\gamma}_2$ tends to 0 as $R \to +\infty$. Because the contour is anticlockwise, we get

$$\widehat{f}(\xi) = -2\pi i \operatorname{Res}\left((\tau+z)^{-k} \mathrm{e}^{-2\pi i z \xi}, -\tau\right) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} \mathrm{e}^{2\pi i \xi \tau}, \xi > 0.$$

Since $f \in \mathfrak{F}$, the Poisson summation formula gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \sum_{m \in \mathbb{Z}} \widehat{f}(m) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \mathrm{e}^{2\pi i m \tau}.$$

The equality obtained above is precisely the desired identity.

(b) Setting $k=2$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = -4\pi^2 \sum_{m=1}^{\infty} m\mathrm{e}^{2\pi i m\tau}.$$

Now $|\mathrm{e}^{2\pi i\tau}| = \mathrm{e}^{-2\pi\operatorname{Im}(\tau)} < 1$, so we have

$$\sum_{m=1}^{\infty} m\mathrm{e}^{2\pi i m\tau} = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\left(\sum_{m=1}^{\infty} \mathrm{e}^{2\pi i m\tau}\right) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{e}^{2\pi i\tau}}{1-\mathrm{e}^{2\pi i\tau}}\right) = \frac{\mathrm{e}^{2\pi i\tau}}{(1-\mathrm{e}^{2\pi i\tau})^2}.$$

Since $1-\mathrm{e}^{2\pi i\tau} = -2i\mathrm{e}^{\pi i\tau}\sin(\pi\tau)$, we can obtain

$$\frac{\mathrm{e}^{2\pi i\tau}}{(1-\mathrm{e}^{2\pi i\tau})^2} = -\frac{1}{4\sin^2(\pi\tau)} \Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Both sides are holomorphic functions of τ on $\mathbb{C}\setminus\mathbb{Z}$. Since they agree on the open set $\operatorname{Im}(\tau)>0$, the Corollary 4.9 of Chapter 2 implies that they agree for all $\tau\in\mathbb{C}\setminus\mathbb{Z}$.

Assignment 8.2 (Exercise 4.4.8)

Suppose $\widehat{f}$ has compact support contained in $[-M,M]$ and let $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Show that

$$a_n = \frac{(2\pi i)^n}{n!}\int_{-M}^{M} \widehat{f}(\xi)\xi^n \mathrm{d}\xi,$$

and as a result

$$\limsup_{n\to\infty} (n!|a_n|)^{1/n} \leqslant 2\pi M.$$

In the converse direction, let f be any power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ with

$$\limsup_{n\to\infty} (n!|a_n|)^{1/n} \leqslant 2\pi M.$$

Then, f is holomorphic in the complex plane, and for every $\varepsilon>0$ there exists $A_\varepsilon>0$ such that

$$|f(z)| \leqslant A_\varepsilon \mathrm{e}^{2\pi(M+\varepsilon)|z|}.$$

Proof Since $\widehat{f}$ has compact support contained in $[-M,M]$, by Fourier inversion formula, we have

$$f(z) = \int_{-M}^{M} \widehat{f}(\xi)\mathrm{e}^{2\pi i z\xi}\mathrm{d}\xi.$$

Expanding the exponential into its power series and interchanging sum and integral, we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{(2\pi i z)^n}{n!}\int_{-M}^{M} \widehat{f}(\xi)\xi^n \mathrm{d}\xi.$$

$$a_n = \frac{(2\pi i)^n}{n!}\int_{-M}^{M} \widehat{f}(\xi)\xi^n \mathrm{d}\xi \Rightarrow |a_n| \leqslant \frac{(2\pi)^n}{n!}\|\widehat{f}\|_{L^1([-M,M])}M^n.$$

Therefore $(n!|a_n|)^{1/n} \leqslant 2\pi M\|\widehat{f}\|_{L^1([-M,M])}^{1/n}$. Letting $n\to\infty$, we get $\limsup_{n\to\infty}(n!|a_n|)^{1/n} \leqslant 2\pi M$. Conversely, suppose $f(z)=\sum_{n=0}^{\infty} a_n z^n$ satisfies $\limsup_{n\to\infty}(n!|a_n|)^{1/n} \leqslant 2\pi M$. Fix $\varepsilon>0$. Then for all sufficiently large n we have $n!|a_n| \leqslant (2\pi(M+\varepsilon))^n$, and after multiplying the RHS by a constant if necessary, this holds for all n. That is, there exists $A_\varepsilon>0$ such that

$$|f(z)| \leqslant A_\varepsilon \sum_{n=0}^{\infty} \frac{(2\pi(M+\varepsilon)|z|)^n}{n!} = A_\varepsilon \mathrm{e}^{2\pi(M+\varepsilon)|z|}.$$

In particular, the radius of convergence is $+\infty$ since $\lim_{n\to\infty}(n!)^{1/n}=+\infty$, so f is entire.

Assignment 8.3 (Exercise 4.4.9)

Here are further results similar to the Phragmén-Lindelöf theorem.

(a) Let F be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(iy)| \leqslant 1$ for all $y \in \mathbb{R}$, and

$$|F(z)| \leqslant C\mathrm{e}^{c|z|^{\gamma}}$$

for some $c, C > 0$ and $\gamma < 1$. Prove that $|F(z)| \leqslant 1$ for all z in the right half-plane.

(b) More generally, let S be a sector whose vertex is the origin, and forming an angle of π/β. Let F be a holomorphic function in S that is continuous on the closure of S, so that $|F(z)| \leqslant 1$ on the boundary of S and

$$|F(z)| \leqslant C\mathrm{e}^{c|z|^{\alpha}} \text{ for all } z \in S$$

for some $c, C > 0$ and $0 < \alpha < \beta$. Prove that $|F(z)| \leqslant 1$ for all $z \in S$.

Proof We only need to prove (b), since (a) is the special case of (b) when $\beta = 1$ and S is the right half-plane. Let $F_\varepsilon(z) = F(z)\mathrm{e}^{-\varepsilon z^r}$, where $r \in (\alpha, \beta) \cap \mathbb{Q}$ and $\varepsilon > 0$. After a rotation, we may assume

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{2\beta} < \arg z < \frac{\pi}{2\beta} \right\}.$$

Then $r \arg z \in (-\pi/2, \pi/2)$ for $z \in S$, so $\cos(r \arg z) > 0$. Hence $|F_\varepsilon(z)| = |F(z)|\mathrm{e}^{-\varepsilon|z|^r \cos(r \arg z)} \leqslant C\mathrm{e}^{c|z|^{\alpha} - \varepsilon|z|^r \cos(r \arg z)}$. Since $r > \alpha$, we have $|F_\varepsilon(z)| \to 0$ as $|z| \to +\infty$ in $\overline{S}$. If F_ε is constant, then $F_\varepsilon = F = 0$. If not, then by the maximum modulus principle, the maximum of $|F_\varepsilon|$ is attained on ∂S. But on ∂S we have $|F(z)| \leqslant 1$, while $\cos(r \arg z) > 0$, so $|F_\varepsilon(z)| \leqslant 1$ for all $z \in S$. Letting $\varepsilon \to 0^+$, we obtain $|F(z)| \leqslant 1$ for all $z \in S$. This proves (b). As noted above, (a) follows by letting $\beta = 1$ and S be the right half-plane. So we are done.

Assignment 8.4 (Exercise 4.4.10)

This exercise generalizes some of the properties of $\mathrm{e}^{-\pi x^2}$ related to the fact that it is its own Fourier transform.

Suppose $f(z)$ is an entire function that satisfies

$$|f(x+iy)| \leqslant c\mathrm{e}^{-ax^2+by^2}$$

for some $a, b, c > 0$. Let

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i x\zeta}\mathrm{d}x.$$

Then, $\widehat{f}$ is an entire function of ζ that satisfies

$$|\widehat{f}(\xi + i\eta)| \leqslant c'\mathrm{e}^{-a'\xi^2+b'\eta^2}$$

for some $a', b', c' > 0$.

[Hint: To prove $\widehat{f}(\xi) = O(\mathrm{e}^{-a'\xi^2})$, assume $\xi > 0$ and change the contour of integration to $x - iy$ for some $y > 0$ fixed, and $-\infty < x < \infty$. Then

$$\widehat{f}(\xi) = O(\mathrm{e}^{-2\pi y\xi}\mathrm{e}^{by^2}).$$

Finally, choose $y = d\xi$ where d is a small constant.]

Proof We first show that $\widehat{f}$ extends to an entire function. For $\zeta = \xi + i\eta$, we have

$$\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i x\zeta}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i x\xi}\mathrm{e}^{2\pi x\eta}\mathrm{d}x.$$

Since $|f(x)| \leqslant c\mathrm{e}^{-ax^2}$ on the real axis, for $\zeta = \xi + i\eta$ in any compact subset of $\mathbb{C}$, the integrand and all its derivatives with respect to ζ are dominated by integrable functions. Hence we can interchange differentiation and integration, thus $\widehat{f}$ is entire. Now fix $\xi \in \mathbb{R}$, we choose $y = d\xi$, where $d > 0$ is sufficiently small such that $bd^2 < 2\pi d$. Then Cauchy's theorem allows us to change the contour of integration from $\mathbb{R}$ to $\mathbb{R} - iy$ by taking the rectangle contour. Therefore

$$\widehat{f}(\xi + i\eta) = \int_{-\infty}^{\infty} f(x - iy)\mathrm{e}^{-2\pi i(x-iy)(\xi+i\eta)}\mathrm{d}x.$$

Hence

$$|\widehat{f}(\xi + i\eta)| \leqslant c\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+by^2}\mathrm{e}^{2\pi x\eta-2\pi y\xi}\mathrm{d}x = c\mathrm{e}^{by^2-2\pi y\xi}\int_{-\infty}^{\infty} \mathrm{e}^{-a\left(x-\frac{\pi\eta}{a}\right)^2+\frac{\pi^2}{a}\eta^2}\mathrm{d}x.$$

Since $y = d\xi$ and $bd^2 < 2\pi d$, we have $by^2 - 2\pi y\xi = (bd^2 - 2\pi d)\xi^2 \triangleq -a'\xi^2$. Therefore, let $b' = \pi^2/a$, we have $|\widehat{f}(\xi + i\eta)| \leqslant c'\mathrm{e}^{-a'\xi^2+b'\eta^2}$ for some $a', b', c' > 0$.

Assignment 8.5 (Exercise 4.4.11)

One can give a neater formulation of the result in Exercise 10 by proving the following fact. Suppose $f(z)$ is an entire function of strict order 2, that is,

$$f(z) = O(\mathrm{e}^{c_1|z|^2})$$

for some $c_1 > 0$. Suppose also that for x real,

$$f(x) = O(\mathrm{e}^{-c_2|x|^2})$$

for some $c_2 > 0$. Then

$$|f(x + iy)| = O(\mathrm{e}^{-ax^2+by^2})$$

for some $a, b > 0$. The converse is obviously true.

Proof Suppose $|f(z)| \leqslant C_1\mathrm{e}^{c_1|z|^2}$ for all $z \in \mathbb{C}$, $|f(x)| \leqslant C_2\mathrm{e}^{-c_2x^2}$ for all $x \in \mathbb{R}$, where $C_1, C_2 > 0$. Choose $\theta \in (0, \pi/2)$ and fix $A \in (0, c_2)$. Let $S_+ = \{z \in \mathbb{C} : 0 < \arg z < \theta\}$. Choose $B > 0$ so large that $c_1 + A\cos(2\theta) - B\sin(2\theta) < 0$. Consider $F(z) = f(z)\mathrm{e}^{(A+iB)z^2}$ on S_+. On the positive real axis we have $|F(x)| = |f(x)|\mathrm{e}^{Ax^2} \leqslant C_2\mathrm{e}^{-(c_2-A)x^2} \leqslant C_2$. On the ray $z = r\mathrm{e}^{i\theta}$, using the growth assumption on f, we get $|F(r\mathrm{e}^{i\theta})| \leqslant C_1\mathrm{e}^{c_1r^2+\mathrm{Re}((A+iB)\mathrm{e}^{2i\theta})r^2} = C_1\mathrm{e}^{(c_1+A\cos(2\theta)-B\sin(2\theta))r^2} \leqslant C_1$. Moreover, F has an order of growth 2, while the angle of S_+ is $\theta < \pi/2$. Hence $2 < \pi/\theta$, then by Assignment 8.3, F is bounded in S_+. Therefore, for $z = x + iy \in S_+$, $|f(z)| \leqslant C\mathrm{e}^{-\mathrm{Re}((A+iB)z^2)} = C\mathrm{e}^{-A(x^2-y^2)+2Bxy} \leqslant C\mathrm{e}^{-Ax^2+Ay^2+2B|xy|} \leqslant C\mathrm{e}^{-Ax^2/2+(A+2B^2/A)y^2}$. The same argument, with B replaced by $-B$, gives the same type of estimate in the sector $S_- = \{z \in \mathbb{C} : -\theta < \arg z < 0\}$. Then, applying the argument above to $f(-z)$ gives the same estimate in the two sectors around the negative real axis. So there exist $a_0, b_0, C_0 > 0$ such that $|f(x + iy)| \leqslant C_0\mathrm{e}^{-a_0x^2+b_0y^2}$ when $z = x + iy$ lies in $\{|\arg z| < \theta\}$ or $\{|\arg z - \pi| < \theta\}$. It remains to consider the complement of these sectors. There we have $|y| \geqslant (\tan\theta)|x|$. Using the global growth estimate, $|f(x + iy)| \leqslant C_1\mathrm{e}^{c_1x^2+c_1y^2}$. Choose $b_1 > 0$ so large that $(b_1 - c_1)y^2 \geqslant (c_1 + a_0)x^2$ when $|y| \geqslant (\tan\theta)|x|$. Then on the complement of these sectors, we have

$c_1x^2 + c_1y^2 \leqslant -a_0x^2 + b_1y^2$, and hence $|f(x+iy)| \leqslant C_1\mathrm{e}^{-a_0x^2+b_1y^2}$. Combining all the estimates, we obtain $|f(x+iy)| = O(\mathrm{e}^{-ax^2+by^2})$ for some $a, b > 0$. The converse is obviously true.

Assignment 8.6 (Exercise 4.4.12)

The principle that a function and its Fourier transform cannot both be too small at infinity is illustrated by the following theorem of Hardy.

If f is a function on $\mathbb{R}$ that satisfies

$$f(x) = O(\mathrm{e}^{-\pi x^2}) \text{ and } \widehat{f}(\xi) = O(\mathrm{e}^{-\pi\xi^2}),$$

then f is a constant multiple of $\mathrm{e}^{-\pi x^2}$. As a result, if $f(x) = O(\mathrm{e}^{-\pi Ax^2})$, and $\widehat{f}(\xi) = O(\mathrm{e}^{-\pi B\xi^2})$, with $AB > 1$ and $A, B > 0$, then f is identically zero.

(a) If f is even, show that $\widehat{f}$ extends to an even entire function. Moreover, if $g(z) = \widehat{f}(z^{1/2})$, then g satisfies

$$|g(x)| \leqslant c\mathrm{e}^{-\pi x} \text{ and } |g(z)| \leqslant c\mathrm{e}^{\pi R\sin^2(\theta/2)} \leqslant c\mathrm{e}^{\pi|z|}$$

when $x \in \mathbb{R}$ and $z = R\mathrm{e}^{i\theta}$ with $R \geqslant 0$ and $\theta \in \mathbb{R}$.

(b) Apply the Phragmén-Lindelöf principle to the function

$$F(z) = g(z)\mathrm{e}^{\gamma z} \text{ where } \gamma = i\pi\frac{\mathrm{e}^{-i\pi/(2\beta)}}{\sin(\pi/(2\beta))}$$

and the sector $0 \leqslant \theta \leqslant \pi/\beta < \pi$, and let $\beta \to 1^+$ to deduce that $\mathrm{e}^{\pi z}g(z)$ is bounded in the closed upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $\mathrm{e}^{\pi z}g(z)$ is constant, as desired.

(c) If f is odd, then $\widehat{f}(0) = 0$, and apply the above argument to $\widehat{f}(z)/z$ to deduce that $f = \widehat{f} = 0$. Finally, write an arbitrary f as an appropriate sum of an even function and an odd function. ♠

Proof (a) Assume that f is even. Then we have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi ix\xi}\mathrm{d}x = 2\int_0^{\infty} f(x)\cos(2\pi x\xi)\mathrm{d}x,$$

so $\widehat{f}$ is even on $\mathbb{R}$. Since $f(x) = O(\mathrm{e}^{-\pi x^2})$, the proof of Assignment 8.4 shows that $\widehat{f}$ extends to an even entire function. Thus its Taylor expansion at 0 contains only even powers. Hence there is an entire function g such that $g(z) = \widehat{f}(z^{1/2})$. $\forall x \geqslant 0$, since $\widehat{f}(\xi) = O(\mathrm{e}^{-\pi\xi^2})$, we obtain $|g(x)| = |\widehat{f}(x^{1/2})| \leqslant c\mathrm{e}^{-\pi x}$, where $c > 0$. On the other hand, from the integral definition of $\widehat{f}$, if $\zeta = \xi + i\eta$, then

$$|\widehat{f}(\zeta)| \leqslant c\int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{e}^{2\pi x\eta}\mathrm{d}x = c\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x-\eta)^2}\mathrm{e}^{\pi\eta^2}\mathrm{d}x = c\mathrm{e}^{\pi\eta^2}.$$

Now let $z = R\mathrm{e}^{i\theta}$ with $R \geqslant 0$ and $\theta \in \mathbb{R}$, let $\zeta = z^{1/2} = \sqrt{R}\mathrm{e}^{i\theta/2}$. Then $(\operatorname{Im}\zeta)^2 = R\sin^2(\theta/2)$, thus we have $|g(z)| = |\widehat{f}(\zeta)| \leqslant c\mathrm{e}^{\pi R\sin^2(\theta/2)} \leqslant c\mathrm{e}^{\pi|z|}$. This proves the required estimates.

(b) Now fix $\beta > 1$ and consider the sector $0 \leqslant \theta \leqslant \pi/\beta < \pi$. Define

$$F(z) = g(z)\mathrm{e}^{\gamma z}, \text{ where } \gamma = i\pi\frac{\mathrm{e}^{-i\pi/(2\beta)}}{\sin(\pi/(2\beta))}.$$

On the boundary ray $\theta = 0$, we have $|g(x)| \leqslant c\mathrm{e}^{-\pi x}$ and $\operatorname{Re}(\gamma x) = \pi x$, hence $|F(x)| \leqslant c$. On the other boundary ray $\theta = \pi/\beta$, the choice of γ gives $\operatorname{Re}(\gamma z) = -\pi|z|$. Together with $|g(z)| \leqslant c\mathrm{e}^{\pi|z|}$, we obtain $|F| \leqslant c$ on both boundary rays. Inside the sector, $F(z)$ has an order of growth $\leqslant 1$, Since

$\beta > 1$, the Phragmén-Lindelöf principle (Assignment 8.3) implies that F is bounded in the sector. Letting $\beta \to 1^+$, we deduce that $e^{\pi z} g(z)$ is bounded in the closed upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem, $e^{\pi z} g(z)$ is constant. Therefore $g(z) = Ce^{-\pi z}$, and hence $\widehat{f}(\xi) = g(\xi^2) = Ce^{-\pi\xi^2}$. Taking the inverse Fourier transform gives $f(x) = Ce^{-\pi x^2}$.
(c) If f is odd. Then $\widehat{f}$ is odd, then $\widehat{f}(0) = 0$. Thus $h(z) = \widehat{f}(z)/z$ has a removable singularity at 0, so it extends to an entire function. Moreover, h is even. Since $\widehat{f}(\xi) = O(e^{-\pi\xi^2})$ on $\mathbb{R}$, we also have $h(\xi) = O(e^{-\pi\xi^2})$ as $|\xi| \to \infty$, and h is bounded near 0. Hence $|h(\xi)| \leqslant Ce^{-\pi\xi^2}$ for all $\xi \in \mathbb{R}$, where $C \geqslant 0$. Now define $G(z) = h(z^{1/2})$, which is entire since h is even. For any $x \geqslant 0$, we have $|G(x)| = |h(x^{1/2})| \leqslant Ce^{-\pi x}$. Also, from the estimate $|\widehat{f}(\zeta)| \leqslant Ce^{\pi(\operatorname{Im}\zeta)^2}$ obtained in (a), and since the singularity of $h(z)$ at 0 is removable, after enlarging the constant we get $|h(\zeta)| \leqslant C'e^{\pi(\operatorname{Im}\zeta)^2}$. So for $z = Re^{i\theta}$ and $\zeta = z^{1/2} = \sqrt{R}e^{i\theta/2}$, we obtain $|G(z)| = |h(\zeta)| \leqslant C'e^{\pi R\sin^2(\theta/2)} \leqslant C'e^{\pi|z|}$. Hence by (b), we have $e^{\pi z}G(z)$ is constant. Hence $G(z) = C''e^{-\pi z}$, so $h(z) = C''e^{-\pi z^2}$, and $\widehat{f}(\xi) = C''\xi e^{-\pi\xi^2}$. Taking the inverse Fourier transform gives $f(x) = \widetilde{C}xe^{-\pi x^2}$. But $f(x) = O(e^{-\pi x^2})$, so $\widetilde{C}$ must be 0. Hence we obtain $f = \widehat{f} = 0$.

Finally, for an arbitrary f, we write $f = f_e + f_o$, where $f_e(x) = (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Then f_e is even and f_o is odd, and both parts satisfy the same decay assumptions. Then by (a), (b) and (c), we have $f_e(x) = Ce^{-\pi x^2}$ and $f_o \equiv 0$. Hence $f(x) = Ce^{-\pi x^2}$, as desired.

To prove the last assertion, assume that $f(x) = O(e^{-\pi Ax^2})$ and $\widehat{f}(\xi) = O(e^{-\pi B\xi^2})$, with $AB > 1$ and $A, B > 0$. Let $h(x) = f(x/\sqrt{A})$. Then $h(x) = O(e^{-\pi x^2})$. Moreover, by the change of variables $u = x/\sqrt{A}$, we have $\widehat{h}(\xi) = \sqrt{A}\,\widehat{f}(\sqrt{A}\,\xi)$, hence $\widehat{h}(\xi) = O(e^{-\pi AB\xi^2}) = O(e^{-\pi\xi^2})$, since $AB > 1$. By the result already proved, $h(x) = Ce^{-\pi x^2}$, so $f(x) = Ce^{-\pi Ax^2}$. Therefore $\widehat{f}(\xi) = CA^{-1/2}e^{-\pi\xi^2/A}$. If $C \neq 0$, then this is not $O(e^{-\pi B\xi^2})$, since $B > 1/A$. Hence $C = 0$ and $f \equiv 0$.

Assignment 8.7 (Exercise 5.6.1)

Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}.$$

[Hint: The function $f/(\psi_{z_1}\cdots\psi_{z_N})$ is nowhere vanishing.]

Proof Let Ω be an open set that contains $\overline{D_R}$, and let f be holomorphic in Ω, with $f(0) \neq 0$ and f vanishes nowhere on C_R. Let $z_1, \cdots, z_N$ be the zeros of f in D_R, counted with multiplicities. First, by setting $\widetilde{f}(z) = f(Rz)$, we can assume $R = 1$ WLOG. Let $g(z) = f(z)/(\psi_{z_1}(z)\cdots\psi_{z_N}(z))$. Since each factor ψ_{z_j} has a simple zero at z_j, each z_j is a removable singularity. Then g is holomorphic in Ω and has no zeros in $\mathbb{D}$. By the same argument as in the textbook, the mean-value property gives

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(e^{i\theta})|\mathrm{d}\theta.$$

Now $|\psi_{z_j}(e^{i\theta})| = 1$ for every $\theta \in \mathbb{R}$, and $\psi_{z_j}(0) = z_j$. Hence we have

$$\log|\psi_{z_j}(0)| = \log|z_j| = \log|z_j| + \frac{1}{2\pi}\int_0^{2\pi} \log|\psi_{z_j}(e^{i\theta})|\mathrm{d}\theta.$$

Now note that if two functions satisfy the hypotheses and the conclusion, then so does their product.

Hence we can obtain Jensen's formula since $f(z) = \psi_{z_1}(z)\cdots\psi_{z_N}(z)g(z)$.

Assignment 8.8 (Exercise 5.6.2)

Find the order of growth of the following entire functions:
(a) $p(z)$ *where* p *is a polynomial.*
(b) e^{bz^n} *for* $b \neq 0$.
(c) $\mathrm{e}^{\mathrm{e}^z}$.

Solution *(a) For every* $\rho > 0$, *we have* $|p(z)| \leqslant A\mathrm{e}^{B|z|^\rho}$ *for all* $z \in \mathbb{C}$ *and some* $A, B > 0$ *since* p *is a polynomial. So the order of growth of* p *is* 0.
(b) Let $f(z) = \mathrm{e}^{bz^n}$, *where* $b \neq 0$. *Since* $|f(z)| = \mathrm{e}^{\mathrm{Re}(bz^n)} \leqslant \mathrm{e}^{|b||z|^n}$, *the order of growth of* f *is at most* n. *Conversely, choose* θ *such that* $b\mathrm{e}^{in\theta} = |b|$. *Then along* $z = r\mathrm{e}^{i\theta}$, *we have* $|f(z)| = \mathrm{e}^{|b|r^n}$. *So the order of growth of* $f(z) = \mathrm{e}^{bz^n}$ *is* n.
(c) Let $f(z) = \mathrm{e}^{\mathrm{e}^z}$. *Along the positive real axis, we have* $|f(r)| = \mathrm{e}^{\mathrm{e}^r}$. *Now for any* $\rho > 0$, *if* f *had an order of growth at most* ρ, *then* $\mathrm{e}^{\mathrm{e}^r} \leqslant A\mathrm{e}^{Br^\rho}$ *for some* $A, B > 0$, *which is impossible as* $r \to +\infty$. *So the order of growth of* $f(z) = \mathrm{e}^{\mathrm{e}^z}$ *is* ∞.

Assignment 8.9 (Exercise 5.6.3)

Show that if τ *is fixed with* $\mathrm{Im}(\tau) > 0$, *then the Jacobi theta function*

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi i n^2 \tau}\mathrm{e}^{2\pi i n z}$$

is of order 2 *as a function of* z. *Further properties of* Θ *will be studied in Chapter 10.*
[Hint: $-n^2t + 2n|z| \leqslant -n^2t/2$ *when* $t > 0$ *and* $n \geqslant 4|z|/t$.*]*

Proof We first show that the order of growth is at most 2. Write $\tau = s + it$ with $t > 0$. Then

$$|\Theta(z|\tau)| \leqslant \sum_{n=-\infty}^{\infty}\left|\mathrm{e}^{\pi i n^2\tau}\mathrm{e}^{2\pi i n z}\right| \leqslant \sum_{n=-\infty}^{\infty}\mathrm{e}^{-\pi n^2 t + 2\pi|n||z|} = -1 + 2\sum_{n=0}^{\infty}\mathrm{e}^{-\pi n^2 t + 2\pi n|z|}.$$

Split the sum into $n < 4|z|/t$ and $n \geqslant 4|z|/t$. If $n < 4|z|/t$, then $2\pi n|z| \leqslant 8\pi|z|^2/t$. If $n \geqslant 4|z|/t$, then $-n^2t + 2n|z| \leqslant -n^2t/2$. Hence we have $|\Theta(z|\tau)| \leqslant C_1\mathrm{e}^{\frac{8\pi}{t}|z|^2} + C_2 \leqslant C\mathrm{e}^{A|z|^2}$. Thus the order of growth of $\Theta(z|\tau)$ is at most 2. Then we use the quasi-periodicity of $\Theta(z|\tau)$ to prove that the order of growth is at least 2, we have $\Theta(z + m\tau|\tau) = \mathrm{e}^{-\pi i m^2\tau - 2\pi i m z}\Theta(z|\tau)$, where $m \in \mathbb{Z}$. Setting $z = 0$ and taking absolute values, we obtain $|\Theta(m\tau|\tau)| = \mathrm{e}^{\pi m^2 t}|\Theta(0|\tau)| = A\mathrm{e}^{B|m\tau|^2}$. Therefore the order of growth cannot be smaller than 2. Hence $\Theta(z|\tau)$ is of order 2 as a function of z.

Assignment 8.10 (Exercise 5.6.4)

Let $t > 0$ *be given and fixed, and define* $F(z)$ *by*

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \mathrm{e}^{-2\pi n t}\mathrm{e}^{2\pi i z}\right).$$

Note that the product defines an entire function of z.
(a) Show that $|F(z)| \leqslant A\mathrm{e}^{a|z|^2}$, *hence* F *has an order of growth* $\leqslant 2$.
(b) F *vanishes exactly when* $z = -int + m$ *for* $n \geqslant 1$ *and* n, m *integers. Thus, if* z_n *is an*

enumeration of these zeros we have

$$\sum \frac{1}{|z_n|^2} = \infty \text{ but } \sum \frac{1}{|z_n|^{2+\varepsilon}} < \infty.$$

[Hint: To prove (a), write $F(z) = F_1(z)F_2(z)$ where

$$F_1(z) = \prod_{n=1}^{N} \left(1 - e^{-2\pi nt}e^{2\pi iz}\right) \text{ and } F_2(z) = \prod_{n=N+1}^{\infty} \left(1 - e^{-2\pi nt}e^{2\pi iz}\right).$$

Choose $N \approx c|z|$ with c appropriately large. Then, since

$$\left(\sum_{N+1}^{\infty} e^{-2\pi nt}\right) e^{2\pi|z|} \leqslant 1,$$

one has $|F_2(z)| \leqslant A$. However,

$$\left|1 - e^{-2\pi nt}e^{2\pi iz}\right| \leqslant 1 + e^{2\pi|z|} \leqslant 2e^{2\pi|z|}.$$

Thus $|F_1(z)| \leqslant 2^N e^{2\pi N|z|} \leqslant e^{c'|z|^2}$. Note that a simple variant of the function F arises as a factor in the triple product formula for the Jacobi theta function Θ, taken up in Chapter 10.]

Proof (a) Given z, choose an integer N such that $|z|/t \leqslant N \leqslant |z|/t + 1$, write $F(z) = F_1(z)F_2(z)$, where $F_1(z) = \prod_{n=1}^{N}(1 - e^{-2\pi nt}e^{2\pi iz})$ and $F_2(z) = \prod_{n=N+1}^{\infty}(1 - e^{-2\pi nt}e^{2\pi iz})$. Since $N \geqslant |z|/t$, we have $e^{-2\pi(N+1)t+2\pi|z|} < 1$. Therefore

$$|F_2(z)| \leqslant \prod_{n=N+1}^{\infty} \left(1 + e^{-2\pi nt}e^{2\pi|z|}\right) \leqslant \prod_{n=N+1}^{\infty} \exp\left(e^{-2\pi nt}e^{2\pi|z|}\right)$$

$$= \exp\left(e^{-2\pi(N+1)t+2\pi|z|}\sum_{m=0}^{\infty} e^{-2\pi mt}\right) \leqslant \exp\left(\frac{1}{1 - e^{-2\pi t}}\right).$$

For F_1, from $N \leqslant |z|/t + 1$ we can obtain

$$|F_1(z)| \leqslant \prod_{n=1}^{N} \left(1 + e^{-2\pi nt}e^{2\pi|z|}\right) \leqslant \left(1 + e^{2\pi|z|}\right)^N \leqslant \left(2e^{2\pi|z|}\right)^N$$

$$\leqslant 2^{|z|/t+1}e^{2\pi|z|(|z|/t+1)} = \exp\left(\log 2 + \left(2\pi + \frac{\log 2}{t}\right)|z| + \frac{2\pi}{t}|z|^2\right)$$

$$\leqslant \exp\left(\left(2\pi + \frac{t+1}{t}\log 2\right) + \left(2\pi\frac{t+1}{t} + \frac{\log 2}{t}\right)|z|^2\right).$$

So $|F(z)| = |F_1(z)||F_2(z)| \leqslant Ae^{a|z|^2}$ for some $A, a > 0$. Hence F has an order of growth $\leqslant 2$.
(b) By Proposition 3.1 of the textbook, $F(z) = 0$ exactly when $1 - e^{-2\pi nt}e^{2\pi iz} = 0$ for some $n \geqslant 1$, i.e., when $z = -int + m$ with $n \geqslant 1$ and $n, m \in \mathbb{Z}$. If $\{z_n\}$ is an enumeration of these zeros, then

$$\sum \frac{1}{|z_n|^2} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{m^2 + n^2t^2} \geqslant \sum_{n=1}^{\infty}\sum_{|m|\leqslant n} \frac{1}{n^2(1+t^2)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(1+t^2)} = \infty.$$

While for every $\varepsilon > 0$, by (a) and Theorem 2.1 of the textbook, we have

$$\sum \frac{1}{|z_n|^{2+\varepsilon}} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(m^2 + n^2t^2)^{1+\varepsilon/2}} < \infty.$$

Moreover, we obtain that the order of growth of F is exactly 2 by Theorem 2.1 of the textbook.

Chapter 9 Homework 9

Assignment 9.1 (Exercise 5.6.11)

Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem.
[Hint: If f misses a, then $f(z) - a$ is of the form $\mathrm{e}^{p(z)}$ where p is a polynomial.]

Proof Suppose that f omits two distinct values a, b. Since f is of finite order and $f - a$ never vanishes, Hadamard's factorization theorem implies that $f(z) - a = \mathrm{e}^{p(z)}$ for some polynomial p. Since f also omits b, we have $\mathrm{e}^{p(z)} \neq b - a$ for all $z \in \mathbb{C}$. If p were nonconstant, then by the fundamental theorem of algebra there would exist $z_0 \in \mathbb{C}$ such that $p(z_0) = \log(b - a)$ for a suitable branch value of the logarithm, equivalently $\mathrm{e}^{p(z_0)} = b - a$, a contradiction. Thus p is constant. Therefore $f(z) = a + \mathrm{e}^{p(z)}$ is also constant. So we are done.

Assignment 9.2 (Exercise 5.6.12)

Suppose f is entire and never vanishes, and that none of the higher derivatives of f ever vanish. Prove that if f is also of finite order, then $f(z) = \mathrm{e}^{az+b}$ for some constants a and b.

Proof Since f is entire, of finite order, and never vanishes, Hadamard's factorization theorem gives $f(z) = \mathrm{e}^{p(z)}$ for some polynomial p. Then $f'(z) = p'(z)\mathrm{e}^{p(z)}$. By our assumption, f' never vanishes, so p' never vanishes. Therefore $\deg p \leqslant 1$. Thus $p(z) = az + b$ for some constants $a, b \in \mathbb{C}$, and consequently $f(z) = \mathrm{e}^{az+b}$. So we are done.

Assignment 9.3 (Exercise 5.6.13)

Show that the equation $\mathrm{e}^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.
[Hint: Apply Hadamard's theorem.]

Proof Define $F(z) = \mathrm{e}^z - z$. This is an entire function of order 1. Suppose that F has only finitely many zeros. By Hadamard's factorization theorem, we can write $F(z) = \mathrm{e}^{az+b}P(z)$, where P is a polynomial. On the positive real axis, $F(x) = \mathrm{e}^x - x$, so $\mathrm{e}^{-x}F(x) \to 1$ as $x \to +\infty$. Hence we have $\mathrm{e}^{(a-1)x+b}P(x) \to 1$ as $x \to +\infty$. This forces P to be constant and $a = 1$. Therefore $F(z) = C\mathrm{e}^z$ for some nonzero constant C. But this is impossible, since $F(z)/\mathrm{e}^z = 1 - z\mathrm{e}^{-z}$ is not constant. Hence F must have infinitely many zeros, so the equation $\mathrm{e}^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

Assignment 9.4 (Exercise 5.6.14)

Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

Proof Suppose that F has only finitely many zeros. By Hadamard's factorization theorem, since F is of finite order, we can write $F(z) = \mathrm{e}^{P(z)}Q(z)$, where P and Q are polynomials. If P is nonconstant, then the order of F is $\deg P$, a positive integer. If P is constant, then F is a polynomial, and its order is 0. In every case, the order of F must be an integer. This contradicts the assumption that the growth order ρ is non-integral. Therefore F has infinitely many zeros.

Assignment 9.5 (Exercise 5.6.15)

Prove that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions. Also, if $\{a_n\}$ and $\{b_n\}$ are two disjoint sequences having no finite limit points, then there exists a meromorphic function in the whole complex plane that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$. ♠

Proof Let f be meromorphic in $\mathbb{C}$, and let $\{p_n\}$ be the sequence of poles of f, counted with multiplicities. Since f is meromorphic, $|p_n| \to +\infty$ as $n \to +\infty$. Thus by Theorem 4.1 of the textbook, there exists an entire function g whose zeros are exactly the points $\{p_n\}$, with the same multiplicities. Therefore $h(z) := f(z)g(z)$ has removable singularities at each point p_n, hence extends to an entire function. Thus $f(z) = h(z)/g(z)$ is the quotient of two entire functions.

For the second assertion, since $\{a_n\}$ and $\{b_n\}$ have no finite limit points, $|a_n| \to +\infty$ and $|b_n| \to +\infty$ as $n \to +\infty$. Thus by Theorem 4.1 of the textbook, we can find entire functions F and G such that F vanishes exactly at the points $\{a_n\}$, and G vanishes exactly at the points $\{b_n\}$, with the same multiplicities. Since the two sequences are disjoint, the quotient $F(z)/G(z)$ is meromorphic in $\mathbb{C}$, has zeros exactly at $\{a_n\}$, and has poles exactly at $\{b_n\}$. So we are done.

Assignment 9.6 (Exercise 5.6.16)

Suppose that

$$Q_n(z) = \sum_{k=1}^{N_n} c_k^n z^k$$

are given polynomials for $n = 1, 2, \cdots$. Suppose also that we are given a sequence of distinct complex numbers $\{a_n\}$ without limit points. Prove that there exists a meromorphic function $f(z)$ whose only poles are at the distinct points $\{a_n\}$, and so that for each n, the difference

$$f(z) - Q_n\left(\frac{1}{z - a_n}\right)$$

is holomorphic near a_n. In other words, f has prescribed poles and principal parts at each of these poles. This result is due to Mittag-Leffler. ♠

Proof Since the sequence $\{a_n\}$ has no finite limit points, we have $|a_n| \to +\infty$ as $n \to +\infty$, and we can add on $Q_n(1/z)$ by hand, so we may assume $a_n \neq 0$. For each n, the function $Q_n(1/(z - a_n))$ is holomorphic in an open neighborhood of the disc $\overline{B(0, |a_n|/2)}$. Its Taylor series at 0 converges uniformly on this disc. Therefore we can choose a polynomial P_n such that

$$\sup_{|z| \leqslant |a_n|/2} \left| Q_n\left(\frac{1}{z - a_n}\right) - P_n(z) \right| \triangleq \sup_{|z| \leqslant |a_n|/2} |f_n(z)| \leqslant \frac{1}{2^n}.$$

For every compact set $K \subset \mathbb{C}$, we have $K \subset \{|z| \leqslant |a_n|/2\}$ for all sufficiently large n. Therefore we can obtain that $\sum_{n=1}^{\infty} f_n$ converges uniformly on K away from the finitely many points a_n that lie in K. Thus $f(z) = \sum_{n=1}^{\infty} f_n(z)$ converges uniformly on every compact subset of $\mathbb{C} \setminus \{a_n\}_{n=1}^{\infty}$ and defines a meromorphic function on $\mathbb{C}$. Near a fixed point a_N, all terms f_n with $n \neq N$ are holomorphic. And the term $f_N(z) = Q_N(1/(z - a_N)) - P_N(z)$, so $f_N(z) - Q_N\left(1/(z - a_N)\right)$ is holomorphic near a_N. Therefore $f(z) - Q_N\left(1/(z - a_N)\right)$ is holomorphic near a_N. And the only poles of $f(z)$ are at distinct points $\{a_n\}$. So we are done.

Assignment 9.7 (Exercise 5.6.17)

Given two arbitrary countably infinite sequences of complex numbers $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$, with $\lim_{k\to\infty}|a_k| = \infty$, it is always possible to find an entire function F that satisfies $F(a_k) = b_k$ for all k.

(a) Given n distinct complex numbers $a_1, \cdots, a_n$, and another n complex numbers $b_1, \cdots, b_n$, construct a polynomial P of degree $\leqslant n-1$ with

$$P(a_i) = b_i \text{ for } i = 1, \cdots, n.$$

(b) Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of distinct complex numbers such that $a_0 = 0$ and $\lim_{k\to\infty}|a_k| = \infty$, and let $E(z)$ denote a Weierstrass product associated with $\{a_k\}$. Given complex numbers $\{b_k\}_{k=0}^{\infty}$, show that there exist integers $m_k \geqslant 1$ such that the series

$$F(z) = \frac{b_0}{E'(0)}\frac{E(z)}{z} + \sum_{k=1}^{\infty}\frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k}\left(\frac{z}{a_k}\right)^{m_k}$$

defines an entire function that satisfies

$$F(a_k) = b_k \text{ for all } k \geqslant 0.$$

This is known as the Pringsheim interpolation formula.

Proof (a) The required polynomial is given by the Lagrange interpolation formula

$$P(z) = \sum_{i=1}^{n} b_i \prod_{j\neq i}\frac{z-a_j}{a_i-a_j}.$$

It has degree $\leqslant n-1$, and substituting $z = a_i$ gives $P(a_i) = b_i$.

(b) Since E is a Weierstrass product associated with $\{a_k\}$, it has simple zeros precisely at the points $\{a_k\}$. Hence $E(z)/(z-a_k)$ extends to an entire function and takes the value $E'(a_k)$ at $z = a_k$. Let

$$T_0(z) = \frac{b_0}{E'(0)}\frac{E(z)}{z},\ T_k(z) = \frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k}\left(\frac{z}{a_k}\right)^{m_k}, k \geqslant 1$$

where $m_k \geqslant 1$ is chosen so large that $|T_k(z)| \leqslant 2^{-k}$ on $\overline{B(0, |a_k|/2)}$. Since $|a_k| \to +\infty$ as $k \to +\infty$, the series $\sum_{k=1}^{\infty} T_k(z)$ converges uniformly on every compact subset of $\mathbb{C}$. Since each T_k is entire, the sum is entire. Adding the entire function $T_0(z)$, we get an entire function F.

It remains to check the interpolation. At $z = 0$, all terms T_k with $k \geqslant 1$ vanish because $m_k \geqslant 1$, and $E(z)/z$ takes the value $E'(0)$ at 0. So $F(0) = b_0$. If $j \geqslant 1$, then at $z = a_j$, all terms with $k \neq j$ vanish because $E(a_j) = 0$, while the j-th term

$$T_j(a_j) = \frac{b_j}{E'(a_j)}E'(a_j)\left(\frac{a_j}{a_j}\right)^{m_j} = b_j.$$

Therefore $F(a_j) = b_j$. Hence we have $F(a_k) = b_k$ for all $k \geqslant 0$.

Chapter 10 Homework 10

Assignment 10.1 (Exercise 5.3.2)

Prove that

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$$

whenever a and b are positive. Using the product formula for $\sin \pi s$, give another proof that $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$.

Proof By Theorem 1.7 of the textbook, for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, we have

$$\Gamma(s+1) = s\Gamma(s) = \mathrm{e}^{-\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} \mathrm{e}^{s/n}.$$

Substituting $s = a, b, a+b$, where $a, b, a+b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, we can obtain

$$\Gamma(a+1) = \mathrm{e}^{-\gamma a} \prod_{n=1}^{\infty} \left(1 + \frac{a}{n}\right)^{-1} \mathrm{e}^{a/n},\ \Gamma(b+1) = \mathrm{e}^{-\gamma b} \prod_{n=1}^{\infty} \left(1 + \frac{b}{n}\right)^{-1} \mathrm{e}^{b/n},$$

$$\Gamma(a+b+1) = \mathrm{e}^{-\gamma(a+b)} \prod_{n=1}^{\infty} \left(1 + \frac{a+b}{n}\right)^{-1} \mathrm{e}^{(a+b)/n}.$$

Since $1/\Gamma(s)$ is entire, the condition $a + b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ can be removed. Therefore, we have

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}, \forall a, b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}.$$

Now take $a = s$ and $b = -s$, where $s \in \mathbb{C} \setminus \mathbb{Z}$. This gives $\Gamma(1+s)\Gamma(1-s) = \prod_{n=1}^{\infty}(1 - s^2/n^2)^{-1}$. Since $\sin \pi s = \pi s \prod_{n=1}^{\infty}(1 - s^2/n^2)$, we get $\Gamma(s)\Gamma(1-s) = \Gamma(1+s)\Gamma(1-s)/s = \pi / \sin \pi s$. And both sides have poles at the integers. So we are done.

Assignment 10.2 (Exercise 5.3.4)

Prove that if we take

$$f(z) = \frac{1}{(1-z)^{\alpha}},\ \textit{for } |z| < 1$$

(defined in terms of the principal branch of the logarithm), where α is a fixed complex number, then

$$f(z) = \sum_{n=0}^{\infty} a_n(\alpha) z^n$$

with

$$a_n(\alpha) \sim \frac{1}{\Gamma(\alpha)} n^{\alpha - 1}\ \textit{as } n \to \infty.$$

Proof First assume that $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$. By the Taylor expansion of $f(z)$ in $\mathbb{D}$, we have

$$f(z) = \frac{1}{(1-z)^{\alpha}} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-z)^n.$$

Hence we can obtain

$$a_n(\alpha) = (-1)^n \binom{-\alpha}{n} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}.$$

So $a_n(\alpha) = \Gamma(n+\alpha)/(\Gamma(\alpha)n!)$. By Exercise 5.3.1, we have

$$\frac{n^{\alpha-1}}{\Gamma(\alpha)} \sim \frac{\alpha(\alpha+1)\cdots(\alpha+n)n^{\alpha-1}}{n^\alpha n!} \sim \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} = a_n(\alpha) \text{ as } n \to \infty.$$

Then if $\alpha \in \mathbb{Z}_{\leqslant 0}$, we have $n^{\alpha-1}/\Gamma(\alpha) = a_n(\alpha) = 0$ for all $n \geqslant -\alpha+1$. So the result is trivial in this case. Hence we only need to prove Exercise 5.3.1.

Assignment (Exercise 5.3.1)

Prove that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}$$

whenever $s \neq 0, -1, -2, \cdots$

[Hint: Use the product formula for $1/\Gamma$*, and the definition of the Euler constant* γ*.]*

Proof By the Weierstrass product formula, for $s \neq 0, -1, -2, \cdots$, we have

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{k=1}^{\infty} \left(1+\frac{s}{k}\right)^{-1} e^{s/k} = \lim_{n\to\infty} \frac{e^{-\gamma s}}{s} \prod_{k=1}^{n} \left(1+\frac{s}{k}\right)^{-1} e^{s/k}.$$

Since

$$\prod_{k=1}^{n} \left(1+\frac{s}{k}\right)^{-1} = \prod_{k=1}^{n} \frac{k}{s+k} = \frac{n!}{(s+1)(s+2)\cdots(s+n)},$$

we can obtain

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! e^{s(1+1/2+\cdots+1/n-\gamma)}}{s(s+1)\cdots(s+n)}.$$

Since

$$\frac{e^{s(1+1/2+\cdots+1/n-\gamma)}}{n^s} = e^{s(1+1/2+\cdots+1/n-\gamma-\log n)} \to 1,$$

we can obtain

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}.$$

This proves the desired formula.

Assignment 10.3 (Exercise 5.3.6)

Show that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\log n \to \frac{\gamma}{2} + \log 2,$$

where γ *is Euler's constant.*

Proof Let $H_n = 1 + 1/2 + \cdots + 1/n$. We use the standard asymptotic formula $H_n = \log n + \gamma + o(1)$, since $1 + 1/3 + \cdots + 1/(2n-1) = H_{2n} - H_n/2$, we have

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\log n &= H_{2n} - \frac{1}{2}H_n - \frac{1}{2}\log n \\ &= \log(2n) + \gamma - \frac{1}{2}(\log n + \gamma) - \frac{1}{2}\log n + o(1) \to \frac{\gamma}{2} + \log 2. \end{aligned}$$

Assignment 10.4 (Exercise 5.3.8)

The Bessel functions arise in the study of spherical symmetries and the Fourier transform. See Chapter 6 in Book I. Prove that the following power series identity holds for Bessel functions of real order $\nu > -1/2$:

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \int_{-1}^{1} \mathrm{e}^{ixt}(1-t^2)^{\nu-(1/2)}\mathrm{d}t = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x^2}{4}\right)^m}{m!\Gamma(\nu+m+1)}$$

whenever $x > 0$. In particular, the Bessel function J_ν satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2 J_\nu}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}J_\nu}{\mathrm{d}x} + \left(1 - \frac{\nu^2}{x^2}\right) J_\nu = 0.$$

[Hint: Expand the exponential e^{ixt} in a power series, and express the remaining integrals in terms of the gamma function, using Exercise 7.]

Proof Since $\nu > -1/2$, and since the power series for e^{ixt} is uniformly absolutely convergent for $-1 \leqslant t \leqslant 1$, we may expand e^{ixt} and integrate term by term. Thus

$$\int_{-1}^{1} \mathrm{e}^{ixt}(1-t^2)^{\nu-1/2}\mathrm{d}t = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \int_{-1}^{1} t^k (1-t^2)^{\nu-1/2}\mathrm{d}t.$$

The odd terms vanish. For $k = 2m$, using $u = t^2$ gives

$$\int_{-1}^{1} t^{2m}(1-t^2)^{\nu-1/2}\mathrm{d}t = \int_0^1 u^{m-1/2}(1-u)^{\nu-1/2}\mathrm{d}u = \frac{\Gamma(\nu+1/2)\Gamma(m+1/2)}{\Gamma(\nu+m+1)}.$$

Therefore the left-hand side in the definition of J_ν becomes

$$\left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \frac{\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(\nu+m+1)}.$$

Since $\Gamma(m+1/2) = (2m)!\sqrt{\pi}/(4^m m!)$, this is exactly

$$\left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x^2}{4}\right)^m}{m!\Gamma(\nu+m+1)}.$$

It remains to check the differential equation. Write

$$J_\nu(x) = \sum_{m=0}^{\infty} c_m x^{\nu+2m}, \quad \text{where } c_m = \frac{(-1)^m}{2^{\nu+2m} m!\Gamma(\nu+m+1)}.$$

Termwise differentiation is allowed by local uniform convergence. The coefficient of $x^{\nu+2m}$ in $J_\nu'' + x^{-1}J_\nu' + (1-\nu^2/x^2)J_\nu$ is $c_m + 4(m+1)(m+\nu+1)c_{m+1} = 0, m \geqslant 0$. And the coefficient of $x^{\nu-2}$ is $\nu(\nu-1)c_0 + \nu c_0 - \nu^2 c_0 = 0$. Hence J_ν satisfies the stated ordinary differential equation.

Assignment 10.5 (Exercise 5.3.9)

The hypergeometric series $F(\alpha,\beta,\gamma;z)$ was defined in Exercise 16 of Chapter 1. Show that

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t.$$

Here $\alpha > 0$, $\beta > 0$, $\gamma > \beta$, and $|z| < 1$.

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1,\infty)$.

Note that

$$\log(1-z) = -zF(1,1,2;z),$$
$$e^z = \lim_{\beta\to\infty} F(1,\beta,1;z/\beta),$$
$$(1-z)^{-\alpha} = F(\alpha,1,1;z).$$

[Hint: To prove the integral identity, expand $(1-zt)^{-\alpha}$ as a power series.]

Proof Since $|z|<1$ and $0\leqslant t\leqslant 1$, we have $|zt|<1$. Hence

$$(1-zt)^{-\alpha} = \sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}z^n t^n.$$

Since $\beta>0$ and $\gamma>\beta$, and since the above expansion is uniformly absolutely convergent for $0\leqslant t\leqslant 1$, substituting this into the integral and integrating term by term gives

$$\text{RHS} = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}z^n\int_0^1 t^{n+\beta-1}(1-t)^{\gamma-\beta-1}\mathrm{d}t.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}z^n\frac{\Gamma(n+\beta)\Gamma(\gamma-\beta)}{\Gamma(n+\gamma)} = \sum_{n=0}^{\infty}\frac{\binom{-\alpha}{n}\binom{-\beta}{n}}{(-1)^n\binom{-\gamma}{n}}z^n = F(\alpha,\beta,\gamma;z).$$

For $z\in\mathbb{C}\setminus[1,\infty)$, the function $1-zt$ does not meet the non-positive real axis for $0\leqslant t\leqslant 1$. Thus the principal branch of $(1-zt)^{-\alpha}$ is single-valued and holomorphic in z on the slit plane $\mathbb{C}\setminus[1,\infty)$. The integral formula therefore gives an analytic continuation of F to this domain.

Finally, the identities $\log(1-z) = -zF(1,1,2;z)$, $(1-z)^{-\alpha} = F(\alpha,1,1;z)$, and $e^z = \lim_{\beta\to\infty} F(\beta,1,1;z/\beta) = \lim_{\beta\to\infty} F(1,\beta,1;z/\beta)$ follow directly from the power series definition of F.

Assignment 10.6 (Exercise 5.3.10)

An integral of the form

$$F(z) = \int_0^{\infty} f(t)t^{z-1}\mathrm{d}t$$

is called a ***Mellin transform****, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t}.*

(a) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^{\infty}\cos(t)t^{z-1}\mathrm{d}t = \Gamma(z)\cos\left(\pi\frac{z}{2}\right) \text{ for } 0<\mathrm{Re}(z)<1,$$

and

$$\mathcal{M}(\sin)(z) = \int_0^{\infty}\sin(t)t^{z-1}\mathrm{d}t = \Gamma(z)\sin\left(\pi\frac{z}{2}\right) \text{ for } 0<\mathrm{Re}(z)<1.$$

(b) Show that the second of the above identities is valid in the larger strip $-1<\mathrm{Re}(z)<1$, and that as a consequence, one has

$$\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2} \text{ and } \int_0^{\infty}\frac{\sin x}{x^{3/2}}\mathrm{d}x = \sqrt{2\pi}.$$

This generalizes the calculation in Exercise 2 of Chapter 2.

[Hint: For the first part, consider the integral of the function $f(w) = e^{-w}w^{z-1}$ around the contour illustrated in Figure 1. Use analytic continuation to prove the second part.]

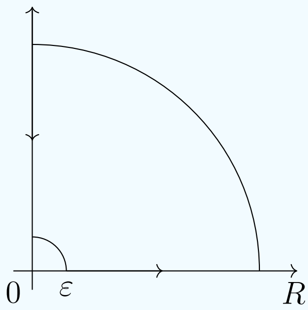

Figure 1. *The contour in Exercise 10*

Proof (a) Take the contour in the hint and choose the principal branch of w^{z-1}. Let $\sigma = \operatorname{Re} z$. On the large arc C_R, write $w = Re^{i\theta}$ with $R > 0$ and $0 \leqslant \theta \leqslant \pi/2$. Then

$$\left|\int_{C_R} \mathrm{e}^{-w} w^{z-1} \mathrm{d}w\right| \leqslant \mathrm{e}^{\pi|\operatorname{Im} z|/2} R^{\sigma} \int_0^{\pi/2} \mathrm{e}^{-R\cos\theta} \mathrm{d}\theta.$$

Since $\cos\theta = \sin(\pi/2-\theta) \geqslant 2(\pi/2-\theta)/\pi$ for $0 \leqslant \theta \leqslant \pi/2$, we have

$$\int_0^{\pi/2} \mathrm{e}^{-R\cos\theta} \mathrm{d}\theta \leqslant \int_0^{\pi/2} \mathrm{e}^{-2R(\pi/2-\theta)/\pi} \mathrm{d}\theta = \frac{\pi}{2R}(1-\mathrm{e}^{-R}) \leqslant \frac{\pi}{2R}.$$

Since $\sigma < 1$, we have

$$\left|\int_{C_R} \mathrm{e}^{-w} w^{z-1} \mathrm{d}w\right| \leqslant \frac{\pi \mathrm{e}^{\pi|\operatorname{Im} z|/2}}{2} R^{\sigma-1} \to 0\,(R \to +\infty).$$

On the small arc C_ε, write $w = \varepsilon \mathrm{e}^{i\theta}$ with $0 < \varepsilon < R$ and $0 \leqslant \theta \leqslant \pi/2$. Since $\sigma > 0$, we have

$$\left|\int_{C_\varepsilon} \mathrm{e}^{-w} w^{z-1} \mathrm{d}w\right| \leqslant \mathrm{e}^{\pi|\operatorname{Im} z|/2} \varepsilon^{\sigma} \int_0^{\pi/2} \mathrm{e}^{-\varepsilon\cos\theta} \mathrm{d}\theta \leqslant \frac{\pi \mathrm{e}^{\pi|\operatorname{Im} z|/2}}{2} \varepsilon^{\sigma} \to 0\,(\varepsilon \to 0^+).$$

Therefore, by Cauchy's theorem, we can obtain

$$\int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t = \int_0^\infty \mathrm{e}^{-it} (it)^{z-1} i \mathrm{d}t.$$

Thus we have

$$\Gamma(z) = i^z \int_0^\infty \mathrm{e}^{-it} t^{z-1} \mathrm{d}t \Rightarrow \int_0^\infty \mathrm{e}^{-it} t^{z-1} \mathrm{d}t = \mathrm{e}^{-i\pi z/2} \Gamma(z).$$

For real z with $0 < z < 1$, taking real and imaginary parts gives

$$\int_0^\infty \cos(t) t^{z-1} \mathrm{d}t = \Gamma(z) \cos\left(\pi \frac{z}{2}\right), \quad \int_0^\infty \sin(t) t^{z-1} \mathrm{d}t = \Gamma(z) \sin\left(\pi \frac{z}{2}\right).$$

Since both sides are holomorphic in $0 < \operatorname{Re} z < 1$, the identities hold throughout this strip.

(b) Firstly, we show that $\mathcal{M}(\sin)(z)$ converges in the strip $-1 < \operatorname{Re} z < 1$. Near 0, since $\sin t \sim t$, the integrand behaves like t^z, which is integrable because $\operatorname{Re} z > -1$. Near $+\infty$, write $g_z(t) = t^{z-1}$, since $\operatorname{Re} z < 1$, $g_z(t) \to 0$ as $t \to +\infty$, and $g_z'(t) = (z-1)t^{z-2}$ is integrable on $[1,\infty)$. Hence, by integration by parts, we can obtain that

$$\int_1^\infty \sin(t) t^{z-1} \mathrm{d}t$$

converges. Similarly, the integral defining $\mathcal{M}(\sin)(z)$ converges uniformly on every compact subset of the strip $-1 < \operatorname{Re} z < 1$. Therefore $\mathcal{M}\sin(z)$ is holomorphic in this strip.

On the other hand, $\Gamma(z)\sin(\pi z/2)$ can extend holomorphically to the same strip $-1 < \operatorname{Re} z < 1$

since $z = 0$ is a removable singularity. Thus by (a) and Corollary 4.9 of Chapter 2, we have

$$\int_0^\infty \sin(t)t^{z-1}\mathrm{d}t = \Gamma(z)\sin\left(\pi\frac{z}{2}\right) \text{ for } -1 < \operatorname{Re} z < 1.$$

Taking $z = 0$, we obtain

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \lim_{z\to 0}\Gamma(z)\sin\frac{\pi z}{2} = \frac{\pi}{2}.$$

Taking $z = -1/2$, we obtain

$$\int_0^\infty \frac{\sin x}{x^{3/2}}\mathrm{d}x = \Gamma\left(-\frac{1}{2}\right)\sin\left(-\frac{\pi}{4}\right) = \sqrt{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}.$$

Assignment 10.7 (Exercise 5.3.11)

Let $f(z) = \mathrm{e}^{az}\mathrm{e}^{-\mathrm{e}^z}$ where $a > 0$. Observe that in the strip $\{x + iy : |y| < \pi/2\}$ the function $f(x + iy)$ is exponentially decreasing as $|x|$ tends to infinity. Prove that

$$\widehat{f}(\xi) = \Gamma(a - 2\pi i\xi), \text{ for all } \xi \in \mathbb{R}.$$

Proof For $z = x + iy$, we have $|f(z)| = \exp(\operatorname{Re}(az - \mathrm{e}^z)) = \exp(ax - \mathrm{e}^x\cos y)$. Thus in the strip $|y| < \pi/2$, the function decreases faster than exponentially as $x \to +\infty$, and decreases exponentially as $x \to -\infty$. So for all real ξ, the Fourier transform exists and is given by

$$\widehat{f}(\xi) = \int_{-\infty}^\infty \mathrm{e}^{ax}\mathrm{e}^{-\mathrm{e}^x}\mathrm{e}^{-2\pi i x\xi}\mathrm{d}x.$$

Let $t = \mathrm{e}^x$. Then $x = \log t$ and $\mathrm{d}x = \mathrm{d}t/t$. Hence

$$\widehat{f}(\xi) = \int_0^\infty t^a\mathrm{e}^{-t}t^{-2\pi i\xi}\frac{\mathrm{d}t}{t} = \int_0^\infty \mathrm{e}^{-t}t^{a-2\pi i\xi-1}\mathrm{d}t = \Gamma(a - 2\pi i\xi).$$

Since $a > 0$, the gamma integral is well-defined for all $\xi \in \mathbb{R}$.

Chapter 11 Homework 11

Assignment 11.1 (Exercise 6.3.13)

Prove that

$$\frac{\mathrm{d}^2 \log \Gamma(s)}{\mathrm{d}s^2} = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, \cdots$

Proof By Theorem 1.7 of the textbook, we have

$$\log \Gamma(s) = -\gamma s - \log s + \sum_{n=1}^{\infty} \left(\frac{s}{n} - \log\left(1 + \frac{s}{n}\right)\right).$$

The expansion is locally uniformly convergent away from $0, -1, -2, \ldots$, so we can differentiate term by term. Thus we have

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+s} \right).$$

Differentiating term by term once more gives

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{1}{(n+s)^2} = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}.$$

For $s > 0$, this is exactly $\mathrm{d}^2 \log \Gamma(s)/\mathrm{d}s^2$. Since both sides are holomorphic on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, the formula holds for all complex numbers s with $s \neq 0, -1, -2, \cdots$

Assignment 11.2 (Exercise 6.3.14)

This exercise gives an asymptotic formula for $\log n!$. A more refined asymptotic formula for $\Gamma(s)$ as $s \to \infty$ (Stirling's formula) is given in Appendix A.

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_x^{x+1} \log \Gamma(t) \mathrm{d}t = \log x, \textit{ for } x > 0,$$

and as a result

$$\int_x^{x+1} \log \Gamma(t) \mathrm{d}t = x \log x - x + c.$$

(b) Show as a consequence that $\log \Gamma(n) \sim n \log n$ as $n \to \infty$. In fact, prove that $\log \Gamma(n) \sim n \log n + O(n)$ as $n \to \infty$. [Hint: Use the fact that $\Gamma(x)$ is monotonically increasing for all large x.]

Proof (a) For $x > 0$, we have $\Gamma(x) > 0$, so by the identity $\Gamma(x+1) = x\Gamma(x)$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_x^{x+1} \log \Gamma(t) \mathrm{d}t = \log \Gamma(x+1) - \log \Gamma(x) = \log x.$$

$$\int_x^{x+1} \log \Gamma(t) \mathrm{d}t = x \log x - x + c.$$

(b) For all sufficiently large t, the function $\log\Gamma(t)$ is increasing. Hence for all large n, we have

$$\log\Gamma(n) \leqslant \int_n^{n+1} \log\Gamma(t)\mathrm{d}t \leqslant \log\Gamma(n+1) = \log n + \log\Gamma(n).$$

Using (a), we can obtain

$$(n-1)\log n - n + c \leqslant \log\Gamma(n) \leqslant n\log n - n + c.$$

Therefore $\log\Gamma(n) = n\log n + O(n)$, and in particular $\log\Gamma(n) \sim n\log n$ as $n \to \infty$.

Assignment 11.3 (Exercise 6.3.17)

Let f be an infinitely differentiable function on $\mathbb{R}$ that has compact support, or more generally, let f belong to the Schwartz space. Consider

$$I(s) = \frac{1}{\Gamma(s)}\int_0^\infty f(x)x^{-1+s}\mathrm{d}x.$$

(a) Observe that $I(s)$ is holomorphic for $\mathrm{Re}(s) > 0$. Prove that I has an analytic continuation as an entire function in the complex plane.

(b) Prove that $I(0) = f(0)$, and more generally

$$I(-n) = (-1)^n f^{(n)}(0) \text{ for all } n \geqslant 0.$$

[Hint: To prove the analytic continuation, as well as the formulas in the second part, integrate by parts to show that $I(s) = \dfrac{(-1)^k}{\Gamma(s+k)}\displaystyle\int_0^\infty f^{(k)}(x)x^{s+k-1}\mathrm{d}x$.]

Proof (a) For $\mathrm{Re}(s) > 0$, the integral defining $I(s)$ is convergent near 0, and the integrand decays rapidly as $x \to \infty$ since f is compactly supported or belongs to the Schwartz space. Hence $I(s)$ is holomorphic for $\mathrm{Re}(s) > 0$. Integrating by parts gives

$$I(s) = \frac{1}{s\Gamma(s)}\int_0^\infty f(x)\mathrm{d}x^s = -\frac{1}{\Gamma(s+1)}\int_0^\infty f'(x)x^s\mathrm{d}x.$$

For every integer $k \geqslant 0$, by applying integration by parts repeatedly, we have

$$I(s) = \frac{(-1)^k}{\Gamma(s+k)}\int_0^\infty f^{(k)}(x)x^{s+k-1}\mathrm{d}x.$$

The RHS is holomorphic for $\mathrm{Re}(s) > -k$. Since $k \in \mathbb{N}^+$ is arbitrary, these formulas give an analytic continuation of I to an entire function in the complex plane.

(b) Taking $k = n+1$ and $s = -n$ in the formula above gives

$$I(-n) = (-1)^{n+1}\int_0^\infty f^{(n+1)}(x)\mathrm{d}x.$$

Since f is compactly supported or rapidly decreasing together with all derivatives, $f^{(n)}(x) \to 0$ as $x \to \infty$. Therefore $I(-n) = (-1)^{n+1}(f^{(n)}(\infty) - f^{(n)}(0)) = (-1)^n f^{(n)}(0)$ for all $n \geqslant 0$.

Assignment 11.4 (Problem 6.4.1)

This problem provides further estimates for ζ and ζ' near $\mathrm{Re}(s) = 1$.

(a) Use Proposition 2.5 and its corollary to prove

$$\zeta(s) = \sum_{1\leqslant n<N} n^{-s} - \frac{N^{1-s}}{1-s} + \sum_{n\geqslant N}\delta_n(s)$$

for every integer $N \geqslant 2$, whenever $\mathrm{Re}(s) > 0$.

(b) Show that $|\zeta(1+it)| = O(\log|t|)$, as $|t| \to \infty$ by using the previous result with N = greatest integer in $|t|$.

(c) The second conclusion of Proposition 2.7 can be similarly refined.

(d) Show that if $t \neq 0$ and t is fixed, then the partial sums of the series $\sum_{n=1}^{\infty} 1/n^{1+it}$ are bounded, but the series does not converge.

Proof (a) Let $\sigma = \mathrm{Re}(s)$. By Proposition 2.5 and its corollary, for each $n \geqslant 1$ we have

$$\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) \mathrm{d}x = \frac{1}{n^s} - \int_n^{n+1} \frac{\mathrm{d}x}{x^s}, \; |\delta_n(s)| \leqslant \frac{|s|}{n^{\sigma+1}}.$$

So $\sum \delta_n(s)$ converges uniformly on every compact subset of $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \sum_{1\leqslant n<N} n^{-s} + \lim_{M\to\infty} \left(\int_N^{M+1} \frac{\mathrm{d}x}{x^s} + \sum_{n=N}^{M} \delta_n(s)\right) = \sum_{1\leqslant n<N} n^{-s} - \frac{N^{1-s}}{1-s} + \sum_{n\geqslant N} \delta_n(s). \; (N \geqslant 2)$$

Both sides are holomorphic on $\{\mathrm{Re}(s) > 0\}$ except at $s = 1$, so the identity holds for $\mathrm{Re}(s) > 0$.

(b) Take $s = 1 + it$ and choose $N = [|t|]$. For sufficiently large $|t|$, from (a), we have

$$|\zeta(1+it)| \leqslant \sum_{1\leqslant n<N} \frac{1}{n} + \frac{1}{|t|} + C|t| \sum_{n\geqslant N} \frac{1}{n^2}.$$

The first term is $O(\log N) = O(\log|t|)$, and the second term is $o(1)$. As for the last term, we have

$$C|t| \sum_{n\geqslant N} \frac{1}{n^2} \leqslant C|t| \left(\frac{1}{N^2} + \int_N^{\infty} \frac{\mathrm{d}x}{x^2}\right) = C|t| \frac{N+1}{N^2} = O(1).$$

Therefore, $|\zeta(1+it)| = O(\log|t|)$ as $|t| \to \infty$.

(c) Differentiating the formula in (a) term by term, we can obtain

$$\zeta'(s) = -\sum_{1\leqslant n<N} \frac{\log n}{n^s} - \frac{-N^{1-s}(1-s)\log N + N^{1-s}}{(1-s)^2} + \sum_{n\geqslant N} \delta_n'(s).$$

And for each $n \geqslant N \geqslant 2$, we have

$$|\delta_n'(s)| \leqslant \int_n^{n+1} \left|\frac{\log n}{x^s} - \frac{\log x}{x^s}\right| \mathrm{d}x \leqslant \sup_{\xi\in[n,n+1]} \left|\frac{\mathrm{d}}{\mathrm{d}z} \frac{\log z}{z^s}\right|_{z=\xi} |n-x| = \sup_{\xi\in[n,n+1]} \left|\frac{1 - s\log\xi}{\xi^{s+1}}\right|$$

$$\leqslant \frac{1 + |s|\log(n+1)}{n^{\sigma+1}} \leqslant C'|s| \frac{\log n}{n^{\sigma+1}}.$$

Taking $s = 1 + it$ and $N = [|t|]$, we get

$$|\zeta'(1+it)| \leqslant \sum_{1\leqslant n<N} \frac{\log n}{n} + \frac{|t|\log N + 1}{|t|^2} + C''|t| \sum_{n\geqslant N} \frac{\log n}{n^2}.$$

The first term is $O(\log^2 N) = O(\log^2|t|)$, and the second term is $o(1)$. And for large N, we have

$$C''|t| \sum_{n\geqslant N} \frac{\log n}{n^2} \leqslant C''|t| \left(\frac{\log N}{N^2} + \int_N^{\infty} \frac{\log x}{x^2} \mathrm{d}x\right) = C''|t| \left(\frac{\log N}{N^2} + \frac{\log N + 1}{N}\right) = O(\log|t|).$$

Therefore, $|\zeta'(1+it)| = O(\log^2|t|)$ as $|t| \to \infty$.

(d) Here $t \in \mathbb{R}$ is necessary; otherwise $t = i$ gives a counterexample. By (a), for $s = 1 + it$, we have

$$\left|\sum_{n=1}^{N-1} \frac{1}{n^{1+it}}\right| = \left|\zeta(1+it) - \frac{N^{-it}}{it} - \sum_{n\geqslant N} \delta_n(1+it)\right| \leqslant |\zeta(1+it)| + \frac{1}{|t|} + \frac{\pi^2 C|t|}{6} < \infty.$$

If the series converged, since $\sum_{n \geqslant N} \delta_n(1+it) \to 0$ as $N \to \infty$, for each fixed $k \in \mathbb{N}^+$, both N^{-it} and $(kN)^{-it}$ would have the same limit, forcing $k^{-it} = 1$ for all $k \in \mathbb{N}^+$. This is impossible for $t \in \mathbb{R}$ and $t \neq 0$. Hence the series does not converge.

Assignment 11.5 (Problem 6.4.2)

Prove that for $\mathrm{Re}(s) > 0$

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} \mathrm{d}x$$

where $\{x\}$ *is the fractional part of* x.

Proof We first prove the identity for $\mathrm{Re}(s) > 1$, we have

$$s \int_1^\infty \frac{\{x\}}{x^{s+1}} \mathrm{d}x = s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{x-n}{x^{s+1}} \mathrm{d}x = s \int_1^\infty \frac{\mathrm{d}x}{x^s} - s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n}{x^{s+1}}.$$

$$= \frac{s}{s-1} - \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \frac{s}{s-1} - \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} + \frac{1}{(n+1)^s} \right)$$

$$= \frac{s}{s-1} - \lim_{N \to \infty} \left(-\frac{1}{(N+1)^{s-1}} + 1 + \sum_{n=2}^{N+1} \frac{1}{n^s} \right) = \frac{s}{s-1} - \zeta(s).$$

Therefore, we can obtain

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} \mathrm{d}x \text{ for } \mathrm{Re}(s) > 1.$$

Both sides are holomorphic on $\{\mathrm{Re}(s) > 0\}$ except for the pole at $s = 1$. Hence the identity holds on the whole half-plane $\{\mathrm{Re}(s) > 0\}$.

Assignment 11.6 (Problem 6.4.3)

If $Q(x) = \{x\} - 1/2$, *then we can write the expression in the previous problem as*

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty \frac{Q(x)}{x^{s+1}} \mathrm{d}x.$$

Let us construct $Q_k(x)$ *recursively so that*

$$\int_0^1 Q_k(x) \mathrm{d}x = 0, \ \frac{\mathrm{d}Q_{k+1}}{\mathrm{d}x} = Q_k(x), \ Q_0(x) = Q(x) \text{ and } Q_k(x+1) = Q_k(x).$$

Then we can write

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} Q_k(x) \right) x^{-s-1} \mathrm{d}x,$$

and a k*-fold integration by parts gives the analytic continuation for* $\zeta(s)$ *when* $\mathrm{Re}(s) > -k$.

Proof By the previous problem and the identity $\{x\} = Q(x) + 1/2$, we get

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{Q(x)+1/2}{x^{s+1}} \mathrm{d}x = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty \frac{Q(x)}{x^{s+1}} \mathrm{d}x.$$

Since $Q_0 = Q$ and $Q_k^{(k)} = Q$, this is the displayed formula

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} Q_k(x) \right) x^{-s-1} \mathrm{d}x.$$

Assume first $\mathrm{Re}(s) > 0$, integration by parts gives

$$\int_1^{\infty} \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} Q_k(x)\right) x^{-s-1}\mathrm{d}x = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} Q_k(x)\right) x^{-s-1}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} Q_k(x)\right) x^{-s-1}\Big|_n^{n+1} + (s+1) \int_n^{n+1} \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} Q_k(x)\right) x^{-s-2}\mathrm{d}x \right]$$

$$= \sum_{n=1}^{\infty} \left(Q_1(x) x^{-s-1}\big|_n^{n+1} \right) + (s+1) \int_1^{\infty} \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} Q_k(x)\right) x^{-s-2}\mathrm{d}x$$

$$= -Q_1(1) + (s+1) \int_1^{\infty} \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} Q_k(x)\right) x^{-s-2}\mathrm{d}x$$

$$= -Q_1(0) + (s+1)\left[-Q_2(0) + (s+2) \int_1^{\infty} \left(\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}} Q_k(x)\right) x^{-s-3}\mathrm{d}x \right]$$

$$= -\sum_{m=1}^{k} Q_m(0) \prod_{j=1}^{m-1} (s+j) + \prod_{j=1}^{k} (s+j) \int_1^{\infty} Q_k(x) x^{-s-k-1}\mathrm{d}x.$$

Substituting this into the previous expression yields

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + s \sum_{m=1}^{k} Q_m(0) \prod_{j=1}^{m-1} (s+j) - s \prod_{j=1}^{k} (s+j) \int_1^{\infty} Q_k(x) x^{-s-k-1}\mathrm{d}x.$$

Since Q_k is bounded on $\mathbb{R}$ by its periodicity, the integral converges for $\mathrm{Re}(s) > -k$, which gives the analytic continuation for $\zeta(s)$ when $\mathrm{Re}(s) > -k$, except for a simple pole at $s = 1$.

Assignment 11.7 (Problem 6.4.4)

The functions Q_k in the previous problem are related to the Bernoulli polynomials $B_k(x)$ by the formula

$$Q_k(x) = \frac{B_{k+1}(x)}{(k+1)!} \text{ for } 0 \leqslant x < 1.$$

Also, if k is a positive integer, then

$$2\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{(2k)!} B_{2k},$$

where $B_k = B_k(0)$ are the Bernoulli numbers. For the definition of $B_k(x)$ and B_k see Chapter 3 in Book I.

Proof Firstly, from Problem 5 of Chapter 3 in Book I, we have

$$B_1(x) = x - 1/2 = Q_0(x), 0 \leqslant x < 1. \; B_n'(x) = nB_{n-1}(x), \int_0^1 B_n(x)\mathrm{d}x = 0, n \geqslant 1.$$

Comparing this with the definition and properties of $\{Q_k\}$ in Assignment 11.6, we obtain

$$Q_k(x) = \frac{B_{k+1}(x)}{(k+1)!}, 0 \leqslant x < 1, k \in \mathbb{N}.$$

It remains to derive the formula for $2\zeta(2k)$. The Fourier series of $Q_0(x) = \{x\} - 1/2$ on $(0, 1)$ is

$$Q_0(x) = -\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n} = -2 \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{2\pi n}, 0 < x < 1.$$

Integrating it $2k-1$ times and using the condition $\displaystyle\int_0^1 Q_k(x)\mathrm{d}x = 0$ at each step, we get

$$Q_{2k-1}(x) = (-1)^{k+1}\frac{2}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}}, 0 < x < 1, k \in \mathbb{N}^+.$$

The series above is absolutely and uniformly convergent on $[0,1]$. So both sides are right-continuous at $x=0$. Letting $x \to 0^+$, we can obtain

$$Q_{2k-1}(0^+) = Q_{2k-1}(0) = (-1)^{k+1}\frac{2}{(2\pi)^{2k}}\zeta(2k).$$

Since $Q_{2k-1}(0) = B_{2k}/(2k)!$, we have

$$\frac{B_{2k}}{(2k)!} = (-1)^{k+1}\frac{2}{(2\pi)^{2k}}\zeta(2k) \Rightarrow 2\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}.$$

Here $k \in \mathbb{N}^+$. This proves the desired formula.

Chapter 12 Homework 12

Assignment 12.1 (Exercise 7.3.4)

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $a_n = a_m$ if $n \equiv m \bmod q$ for some positive integer q. Define the ***Dirichlet*** ***L-series*** *associated to $\{a_n\}$ by*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ for } \operatorname{Re}(s) > 1.$$

Also, with $a_0 = a_q$, let

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} \mathrm{e}^{mx}.$$

Show, as in Exercises 15 and 16 of the previous chapter, that

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{\mathrm{e}^{qx}-1} \mathrm{d}x \text{ for } \operatorname{Re}(s) > 1.$$

Prove as a result that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s=1$. In fact, $L(s)$ is regular at $s=1$ if and only if $\sum_{m=0}^{q-1} a_m = 0$. Note the connection with the Dirichlet $L(s,\chi)$ series, taken up in Book I, Chapter 8, and that as a consequence, $L(s,\chi)$ is regular at $s=1$ if and only if χ is a non-trivial character.

Proof Firstly, for $x>0$, we have

$$\frac{Q(x)}{\mathrm{e}^{qx}-1} = \left(\sum_{m=0}^{q-1} a_{q-m}\mathrm{e}^{mx}\right)\left(\sum_{k=1}^{\infty} \mathrm{e}^{-kqx}\right) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-nx}$$

since $a_n = a_m$ whenever $n \equiv m \bmod q$. Therefore, for $\operatorname{Re}(s) > 1$, absolute convergence allows us to interchange summation and integration, and we get

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x)x^{s-1}}{\mathrm{e}^{qx}-1}\mathrm{d}x = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} a_n \int_0^{\infty} \mathrm{e}^{-nx}x^{s-1}\mathrm{d}x = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = L(s).$$

Choose $\delta > 0$ small, then we have

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\delta} \frac{Q(x)x^{s-1}}{\mathrm{e}^{qx}-1}\mathrm{d}x + \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} \frac{Q(x)x^{s-1}}{\mathrm{e}^{qx}-1}\mathrm{d}x.$$

The second integral defines an entire function of s, because the integrand decays exponentially as $x \to \infty$. For the first integral, write

$$H(x) = \frac{qxQ(x)}{\mathrm{e}^{qx}-1}.$$

Then H is holomorphic near 0, and $H(0) = Q(0) = \sum_{m=0}^{q-1} a_m$. For any $N \in \mathbb{N}^+$, we write $H(x) = \sum_{k=0}^{N-1} c_k x^k + x^N R_N(x)$ near 0. Then

$$\frac{1}{\Gamma(s)} \int_0^{\delta} \frac{Q(x)x^{s-1}}{\mathrm{e}^{qx}-1}\mathrm{d}x = \frac{1}{q\Gamma(s)} \left(\sum_{k=0}^{N-1} \frac{c_k\delta^{s+k-1}}{s+k-1} + \int_0^{\delta} R_N(x)x^{s+N-2}\mathrm{d}x\right).$$

This gives a meromorphic continuation to $\{\operatorname{Re}(s) > 1-N\}$. And the possible poles other than $s=1$ are canceled by the zeros of $1/\Gamma(s)$. Hence $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s=1$, and the coefficient of the pole term $1/(s-1)$ is $c_0\delta^{s-1}/(q\Gamma(s))$, whose value at $s=1$ is c_0/q. Therefore $L(s)$ is regular at $s=1$ if and only if $c_0=H(0)=Q(0)=\sum_{m=0}^{q-1}a_m=0$. We can obtain $L(s,\chi)$ by taking $a_n=\chi(n)$. Since $\sum_{m=0}^{q-1}\chi(m)=0$ if and only if χ is non-trivial, it follows that $L(s,\chi)$ is regular at $s=1$ if and only if χ is a non-trivial character.

Assignment 12.2 (Exercise 7.3.7)

Show that the function

$$\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is real when s is real, or when $\mathrm{Re}(s)=1/2$.

Proof When $s\in\mathbb{R}$, the conclusion is obvious. When $\mathrm{Re}(s)=1/2$, write $s=1/2+it$, by the proof of Theorem 2.3 of Chapter 6, we have

$$\xi(s)=\frac{1}{s-1}-\frac{1}{s}+\int_1^\infty(u^{-s/2-1/2}+u^{s/2-1})\psi(u)\mathrm{d}u$$

where $\psi(u)=(\vartheta(u)-1)/2=\sum_{n=1}^\infty \mathrm{e}^{-\pi n^2u}$. Substituting $s=1/2+it$ into the integral gives

$$\int_1^\infty(u^{-3/4-it/2}+u^{-3/4+it/2})\psi(u)\mathrm{d}u\in\mathbb{R}.$$

And $1/(-1/2+it)$ and $1/(-1/2-it)$ are also conjugate to each other. Therefore $\xi(1/2+it)\in\mathbb{R}$.

Assignment 12.3 (Exercise 7.3.8)

The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

(a) Let

$$F(s)=\xi(1/2+s),\ \textit{where}\ \xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Show that $F(s)$ is an even function of s, and as a result, there exists G so that $G(s^2)=F(s)$.

(b) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is,

$$|(s-1)\zeta(s)|\leqslant A_\varepsilon \mathrm{e}^{a_\varepsilon|s|^{1+\varepsilon}}.$$

As a consequence $G(s)$ is of growth order $1/2$.

(c) Deduce from the above that ζ has infinitely many zeros in the critical strip.

[Hint: To prove (a) and (b) use the functional equation for $\zeta(s)$. As for (c), use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5).]

Proof (a) By the functional equation $\xi(s)=\xi(1-s)$, we have $F(-s)=\xi(1/2-s)=\xi(1-(1/2+s))=\xi(1/2+s)=F(s)$. Thus F is an even function. Consequently its Laurent expansion contains only even powers of s, so there exists a meromorphic function G such that $G(s^2)=F(s)$.

(b) We first estimate the order of $\xi(s)$. Let $\sigma=\mathrm{Re}(s)$, then by the proof of Theorem 2.3 of Chapter 6, we can obtain

$$|\xi(s)|=\left|\frac{1}{s-1}-\frac{1}{s}+\int_1^\infty\left(u^{-s/2-1/2}+u^{s/2-1}\right)\psi(u)\mathrm{d}u\right|$$

$$\leqslant\left|\frac{1}{s-1}\right|+\left|\frac{1}{s}\right|+\int_1^\infty\left(u^{-\sigma/2-1/2}+u^{\sigma/2-1}\right)\left(\sum_{n=1}^\infty \mathrm{e}^{-\pi n^2u}\right)\mathrm{d}u$$

$$\leqslant \left|\frac{1}{s-1}\right| + \left|\frac{1}{s}\right| + 2\sum_{n=1}^{\infty}\int_0^{\infty} u^{|\sigma|/2+1}\mathrm{e}^{-\pi n^2 u}\mathrm{d}u \xlongequal{t=\pi n^2 u} \left|\frac{1}{s-1}\right| + \left|\frac{1}{s}\right| + 2\sum_{n=1}^{\infty}\int_0^{\infty} \frac{t^{|\sigma|/2+1}}{(\pi n^2)^{|\sigma|/2+2}}\mathrm{e}^{-t}\mathrm{d}t$$

$$= \left|\frac{1}{s-1}\right| + \left|\frac{1}{s}\right| + 2\Gamma\left(\frac{|\sigma|}{2}+2\right)\sum_{n=1}^{\infty}\frac{1}{(\pi n^2)^{|\sigma|/2+2}} \leqslant c\cdot\Gamma\left(\frac{|\sigma|}{2}+2\right)$$

when $|s|$ is sufficiently large. By Theorem 2.3 of Appendix A, we have $\Gamma(x) \leqslant C\mathrm{e}^{x\log x}$ for sufficiently large $x > 0$. Thus we can obtain $|\xi(s)| \leqslant A'_\varepsilon \mathrm{e}^{a'_\varepsilon |\sigma|^{1+\varepsilon}} \leqslant A'_\varepsilon \mathrm{e}^{a'_\varepsilon |s|^{1+\varepsilon}}$ when $|s|$ is sufficiently large. Since $(s-1)\zeta(s) = \pi^{s/2}(s-1)\xi(s)/\Gamma(s/2)$, and the functions $\pi^{s/2}$ and $1/\Gamma(s/2)$ have growth order 1 and at most 1 respectively, hence $(s-1)\zeta(s)$ is an entire function of growth order 1. Since $F(s) = \xi(1/2+s)$, the function $(s^2-1/4)F(s)$ is entire of growth order 1. While $(s^2-1/4)F(s) = (s^2-1/4)G(s^2)$, thus $(s-1/4)G(s)$ is entire of growth order $1/2$. Hence $G(s)$ is of growth order $1/2$.

(c) Applying Hadamard's theorem to $(s-1/4)G(s)$, we see that it has infinitely many zeros. So G and hence F and ξ have infinitely many zeros. Outside the critical strip, ξ has no zeros: in $\mathrm{Re}(s) > 1$ this follows from the Euler product for ζ and $\pi^{-s/2}\Gamma(s/2) \neq 0$, while in $\mathrm{Re}(s) < 0$ it follows from the functional equation $\xi(s) = \xi(1-s)$. Thus ζ has infinitely many zeros in the critical strip since $\pi^{-s/2}$ has no zeros and $1/\Gamma(s/2)$ is entire. So we are done.

Assignment 12.4 (Exercise 7.3.10)

In the theory of primes, a better approximation to $\pi(x)$ (instead of $x/\log x$) turns out to be $\mathrm{Li}(x)$ defined by

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

(a) Prove that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \text{ as } x \to \infty,$$

and that as a consequence

$$\pi(x) \sim \mathrm{Li}(x) \text{ as } x \to \infty.$$

[Hint: Integrate by parts in the definition of $\mathrm{Li}(x)$ and observe that it suffices to prove

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = O\left(\frac{x}{(\log x)^2}\right).$$

To see this, split the integral from 2 to $\sqrt{x}$ and from $\sqrt{x}$ to x.]

(b) Refine the previous analysis by showing that for every integer $N > 0$ one has the following asymptotic expansion

$$\mathrm{Li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \cdots + (N-1)!\frac{x}{(\log x)^N} + O\left(\frac{x}{(\log x)^{N+1}}\right)$$

as $x \to \infty$.

Proof (a) Integrating by parts in the definition of $\mathrm{Li}(x)$ gives

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{\mathrm{d}t}{(\log t)^2}.$$

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^{\sqrt{x}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{(\log t)^2} \leqslant \frac{\sqrt{x}-2}{(\log 2)^2} + \frac{4(x-\sqrt{x})}{(\log x)^2} = O(\sqrt{x}) + O\left(\frac{x}{(\log x)^2}\right)$$

as $x \to \infty$. Since $\sqrt{x} = O(x/(\log x)^2)$, we can obtain

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = O\left(\frac{x}{(\log x)^2}\right) \Rightarrow \mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \text{ as } x \to \infty.$$

Since $\pi(x) \sim x/\log x$ as $x \to \infty$, it follows that $\pi(x) \sim \mathrm{Li}(x)$ as $x \to \infty$.

(b) For every integer $N > 0$, repeated integration by parts gives

$$\mathrm{Li}(x) = \left(\frac{t}{\log t} + \frac{t}{(\log t)^2} + 2\frac{t}{(\log t)^3} + \cdots + (N-1)!\frac{t}{(\log t)^N}\right)\Bigg|_2^x + N!\int_2^x \frac{\mathrm{d}t}{(\log t)^{N+1}}.$$

As in (a), splitting the last integral at $\sqrt{x}$ gives

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^{N+1}} = O\left(\frac{x}{(\log x)^{N+1}}\right) \text{ as } x \to \infty.$$

Therefore we can obtain the asymptotic expansion

$$\mathrm{Li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \cdots + (N-1)!\frac{x}{(\log x)^N} + O\left(\frac{x}{(\log x)^{N+1}}\right) \text{ as } x \to \infty.$$

Assignment 12.5 (Exercise 7.3.11)

Let

$$\varphi(x) = \sum_{p \leqslant x} \log p$$

where the sum is taken over all primes $\leqslant x$. Prove that the following are equivalent as $x \to \infty$:

(i) $\varphi(x) \sim x$,

(ii) $\pi(x) \sim x/\log x$,

(iii) $\psi(x) \sim x$,

(iv) $\psi_1(x) \sim x^2/2$.

Proof Firstly, the functions appearing in the problem are

$$\varphi(x) = \sum_{p \leqslant x} \log p,\ \pi(x) = \sum_{p \leqslant x} 1,\ \psi(x) = \sum_{p^m \leqslant x} \log p,\ \psi_1(x) = \int_1^x \psi(u)\mathrm{d}u.$$

(i) $\Leftrightarrow$ (ii): For any $0 < \alpha < 1$, we have

$$\alpha \log x(\pi(x) - x^\alpha) \leqslant \log x^\alpha(\pi(x) - \pi(x^\alpha)) \leqslant \varphi(x) \leqslant \psi(x) \leqslant \pi(x)\log x. \quad (\star)$$

If $\varphi(x) \sim x$, then these inequalities give $1 \leqslant \liminf_{x\to\infty} \pi(x)\log x/x \leqslant \limsup_{x\to\infty} \pi(x)\log x/x \leqslant 1/\alpha$. Letting $\alpha \to 1^-$ gives $\pi(x) \sim x/\log x$. Conversely, if $\pi(x) \sim x/\log x$, the same inequalities give $\alpha \leqslant \liminf_{x\to\infty} \varphi(x)/x \leqslant \limsup_{x\to\infty} \varphi(x)/x \leqslant 1$. Letting $\alpha \to 1^-$ gives $\varphi(x) \sim x$.

(ii) $\Leftrightarrow$ (iii): By $(\star)$, for any $0 < \alpha < 1$, we have

$$\frac{\alpha \log x\,(\pi(x) - x^\alpha)}{x} \leqslant \frac{\psi(x)}{x} \leqslant \frac{\pi(x)\log x}{x}.$$

The rest of the argument is completely analogous to the proof of (i) $\Leftrightarrow$ (ii).

(iii) $\Leftrightarrow$ (iv): For all α, β satisfying $0 < \alpha < 1 < \beta$, we have

$$\frac{1}{1-\alpha}\left(\frac{\psi_1(x)}{x^2} - \frac{\psi_1(\alpha x)}{(\alpha x)^2}\alpha^2\right) \leqslant \frac{\psi(x)}{x} \leqslant \frac{1}{\beta - 1}\left(\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right).$$

If $\psi(x) \sim x$, then we have

$$\frac{1}{\beta+1} \leqslant \liminf_{x\to\infty} \frac{\psi_1(x)}{x^2} \leqslant \limsup_{x\to\infty} \frac{\psi_1(x)}{x^2} \leqslant \frac{1}{1+\alpha}.$$

Letting $\alpha \to 1^-$ and $\beta \to 1^+$ gives $\psi_1(x) \sim x^2/2$. And if $\psi_1(x) \sim x^2/2$, then we have

$$\frac{1+\alpha}{2} \leqslant \liminf_{x\to\infty} \frac{\psi(x)}{x} \leqslant \limsup_{x\to\infty} \frac{\psi(x)}{x} \leqslant \frac{\beta+1}{2}.$$

Letting $\alpha \to 1^-$ and $\beta \to 1^+$ gives $\psi(x) \sim x$.

Assignment 12.6 (Exercise 7.3.12)

If p_n denotes the n^{th} prime, the prime number theorem implies that $p_n \sim n \log n$ as $n \to \infty$.
(a) Show that $\pi(x) \sim x/\log x$ implies that

$$\log \pi(x) + \log\log x \sim \log x.$$

(b) As a consequence, prove that $\log \pi(x) \sim \log x$, and take $x = p_n$ to conclude the proof. ♠

Proof (a) Since $\pi(x) \sim x/\log x$, we have $\pi(x)\log x/x \to 1$. Taking logarithms gives $\log \pi(x) + \log\log x - \log x \to 0$. Therefore $\log \pi(x) + \log\log x \sim \log x$.
(b) Since $\log\log x/\log x \to 0$, (a) implies $\log \pi(x) \sim \log x$. Taking $x = p_n$, we have $\pi(p_n) = n$, and hence $\log n \sim \log p_n$. Also $\pi(x)\log x/x \to 1$ gives $\pi(p_n)\log p_n/p_n = n\log p_n/p_n \to 1$. Combining this with $\log n/\log p_n \to 1$, we obtain $n\log n/p_n \to 1$. Thus $p_n \sim n\log n$ as $n \to \infty$.

Assignment 12.7 (Problem 7.4.1)

Let $F(s) = \sum_{n=1}^{\infty} a_n/n^s$, where $|a_n| \leqslant M$ for all n.
(a) Then

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |F(\sigma+it)|^2 \mathrm{d}t = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \text{ if } \sigma > 1.$$

How is this reminiscent of the Parseval-Plancherel theorem? See e.g. Chapter 3 in Book I.
(b) Show as a consequence the uniqueness of Dirichlet series: If $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where the coefficients are assumed to satisfy $|a_n| \leqslant Cn^k$ for some k, and $F(s) = 0$, then $a_n = 0$ for all n.
Hint: For part (a) use the fact that

$$\frac{1}{2T}\int_{-T}^{T} (nm)^{-\sigma} n^{-it} m^{it} \mathrm{d}t \to \begin{cases} n^{-2\sigma}, & n = m, \\ 0, & n \neq m. \end{cases}$$

♠

Proof (a) Since $\sigma > 1$ and $|a_n| \leqslant M$ for all $n \geqslant 1$, the series defining $F(\sigma+it)$ converges absolutely and uniformly in t. Let $F_N(s) = \sum_{n=1}^{N} a_n/n^s$, where $N \geqslant 1$. Then we have

$$|F_N(\sigma+it)|^2 = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{a_n\overline{a_m}}{(nm)^\sigma} \mathrm{e}^{it\log(m/n)}.$$

Therefore, we can obtain

$$\frac{1}{2T}\int_{-T}^{T} |F_N(\sigma+it)|^2 \mathrm{d}t = \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{1\leqslant n,m\leqslant N, n\neq m} \frac{a_n\overline{a_m}}{(nm)^\sigma} \frac{1}{2T}\int_{-T}^{T} \mathrm{e}^{it\log(m/n)} \mathrm{d}t.$$

By the fact in the hint, the second term is equal to

$$\sum_{n\neq m} a_n\overline{a_m} \cdot \frac{1}{2T}\int_{-T}^{T} (nm)^{-\sigma} n^{-it} m^{it} \mathrm{d}t = \sum_{n\neq m} a_n\overline{a_m} \frac{\sin(\log(m/n)T)}{(nm)^\sigma \log(m/n)T} \to 0 \text{ as } T \to \infty.$$

Therefore, we can obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F_N(\sigma+it)|^2 \mathrm{d}t = \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}}.$$

Now let $N \to \infty$. Since $F_N \to F$ uniformly on the line $\mathrm{Re}(s) = \sigma$, we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma+it)|^2 \mathrm{d}t = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}.$$

This is analogous to the Parseval-Plancherel theorem, with the functions $n^{-it} = \mathrm{e}^{-it\log n}$ playing the role of mutually orthogonal exponentials in the limiting mean.

(b) Suppose $F(s) = \sum_{n=1}^{\infty} a_n n^{-s} = 0$, and $|a_n| \leqslant Cn^k$. Let $c_n = a_n/n^k$. Then $|c_n| \leqslant C$, and

$$F(s+k) = \sum_{n=1}^{\infty} c_n n^{-s}.$$

For every $\sigma > 1$, applying (a) to this Dirichlet series gives

$$0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma+k+it)|^2 \mathrm{d}t = \sum_{n=1}^{\infty} \frac{|c_n|^2}{n^{2\sigma}}.$$

Hence $c_n = 0$ for all $n \geqslant 1$, so $a_n = 0$ for all $n \geqslant 1$. This proves the uniqueness of Dirichlet series.

Chapter 13 Homework 13

Assignment 13.1 (Exercise 8.5.2)

Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0, and so that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

[Hint: Write $F(z) = (g(z))^2$ for z near z_0, and consider the mapping $z \mapsto g(z)$ and its inverse.] ♠

Proof Since $F(z_0) = F'(z_0) = 0$ and $F''(z_0) \neq 0$, the zero of F at z_0 has order exactly 2. Therefore, near z_0, we may write $F(z) = (g(z))^2$, where $g(z)$ is holomorphic and $g(z) = c(z - z_0) + \cdots$, with $c \neq 0$. Thus $g'(z_0) \neq 0$. The inverse function theorem gives neighbourhoods U of z_0 and V of 0 such that $g|_U : U \to V$ is biholomorphic. We choose $\varepsilon > 0$ sufficiently small such that $B(0, \varepsilon) \subset V$. Now let $\Gamma_1 = (g|_U)^{-1}((-\varepsilon, \varepsilon))$, $\Gamma_2 = (g|_U)^{-1}(i(-\varepsilon, \varepsilon))$. Then both curves pass through z_0. Since $g|_U$ is biholomorphic, $(g|_U)^{-1}$ preserves angles, so Γ_1 and Γ_2 are orthogonal at z_0. On Γ_1, g is real-valued, and hence $F = g^2 \geqslant 0$, with equality only at z_0. So F has a minimum on Γ_1 at z_0. On Γ_2, g is purely imaginary, and hence $F = g^2 \leqslant 0$, with equality only at z_0. So F has a maximum on Γ_2 at z_0.

Assignment 13.2 (Exercise 8.5.4)

Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$?

[Hint: Move the upper half-plane "down" and then square it to get $\mathbb{C}$.] ♠

Solution *Yes. Consider the holomorphic function on $\mathbb{D}$ defined by*

$$H(z) = \left(i\frac{1-z}{1+z} - i\right)^2 = (G(z) - i)^2 = \frac{-4z^2}{(1+z)^2}.$$

Since $G(z) = i(1-z)/(1+z)$ is a biholomorphic map from $\mathbb{D}$ to $\mathbb{H}$, $G(z) - i$ is a biholomorphic map from $\mathbb{D}$ to $\{w \in \mathbb{C} : \operatorname{Im} w > -1\} \triangleq S$. For any $w \in \mathbb{C}$, choose η such that $\eta^2 = w$ and $\operatorname{Im} \eta \geqslant 0$. Then $\eta \in S$, so $\exists\, z \in \mathbb{D}$ such that $G(z) - i = \eta$. Thus $H(z) = w$. So H is a holomorphic surjection from $\mathbb{D}$ to $\mathbb{C}$. This proves the desired result.

Assignment 13.3 (Exercise 8.5.7)

Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

(a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}.$$

This leads to two separate cases: either $0 < y \leqslant 1/2$ and $\theta = \pi/2$, or $1/2 \leqslant y < 1$ and $\theta = -\pi/2$. In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \text{ and } P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}.$$

(b) In the integral $\dfrac{1}{2\pi}\displaystyle\int_0^{\pi} P_r(\theta-\varphi)\widetilde{f}_0(\varphi)\mathrm{d}\varphi$ *make the change of variables* $t = F(\mathrm{e}^{i\varphi})$. *Observe that*

$$\mathrm{e}^{i\varphi} = \frac{i-\mathrm{e}^{\pi t}}{i+\mathrm{e}^{\pi t}},$$

and then take the imaginary part and differentiate both sides to establish the two identities

$$\sin\varphi = \frac{1}{\cosh \pi t} \text{ and } \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\pi}{\cosh \pi t}.$$

Hence deduce that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi} P_r(\theta-\varphi)\widetilde{f}_0(\varphi)\mathrm{d}\varphi &= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\widetilde{f}_0(\varphi)\mathrm{d}\varphi\\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty}\frac{f_0(t)}{\cosh\pi t-\cos\pi y}\mathrm{d}t.\end{aligned}$$

(c) Use a similar argument to prove the formula for the integral $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{0} P_r(\theta-\varphi)\widetilde{f}_1(\varphi)\mathrm{d}\varphi$.

Proof (a) From the definition of the conformal map G, we have

$$G(iy) = \frac{i-\mathrm{e}^{i\pi y}}{i+\mathrm{e}^{i\pi y}} = \frac{-\cos\pi y + i(1-\sin\pi y)}{\cos\pi y + i(1+\sin\pi y)} = i\frac{\cos\pi y}{1+\sin\pi y}.$$

Thus, if $G(iy) = r\mathrm{e}^{i\theta}$, then for $0<y\leqslant 1/2$ we have $\theta = \pi/2$ and $r = \cos\pi y/(1+\sin\pi y)$, while for $1/2 \leqslant y < 1$ we have $\theta = -\pi/2$ and $r = -\cos\pi y/(1+\sin\pi y)$. In both cases, we have

$$r^2 = \left(\frac{\cos\pi y}{1+\sin\pi y}\right)^2 = \frac{1-\sin\pi y}{1+\sin\pi y}.$$

Therefore, we can obtain

$$\begin{aligned}P_r(\theta-\varphi) &= \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} = \frac{\sin\pi y}{1-r(1+\sin\pi y)\cos(\theta-\varphi)}\\ &= \frac{\sin\pi y}{1-(\pm\cos\pi y)(\pm\sin\varphi)} = \frac{\sin\pi y}{1-\cos\pi y\sin\varphi}.\end{aligned}$$

(b) Let $t = F(\mathrm{e}^{i\varphi})$, then $\mathrm{e}^{\pi t} = i(1-\mathrm{e}^{i\varphi})/(1+\mathrm{e}^{i\varphi})$, hence $\mathrm{e}^{i\varphi} = (i-\mathrm{e}^{\pi t})/(i+\mathrm{e}^{\pi t})$ and $\widetilde{f}_0(\varphi) = f_0(t)$. And as φ travels from 0 to π, then t goes from $-\infty$ to $+\infty$ on the real line. Taking the imaginary parts of the identity and differentiating the identity give

$$\sin\varphi = \frac{1}{\cosh\pi t},\ \mathrm{d}\varphi = \frac{-\pi\mathrm{e}^{\pi t}(i+\mathrm{e}^{\pi t}) - \pi\mathrm{e}^{\pi t}(i-\mathrm{e}^{\pi t})}{i\mathrm{e}^{i\varphi}(i+\mathrm{e}^{\pi t})^2}\mathrm{d}t = \frac{\pi}{\cosh\pi t}\mathrm{d}t.$$

Therefore, we can obtain

$$\frac{1}{2\pi}\int_0^{\pi} P_r(\theta-\varphi)\widetilde{f}_0(\varphi)\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\widetilde{f}_0(\varphi)\mathrm{d}\varphi = \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_0(t)}{\cosh\pi t-\cos\pi y}\mathrm{d}t.$$

(c) We make the change of variables $\mathrm{e}^{i\varphi} = (i+\mathrm{e}^{\pi t})/(i-\mathrm{e}^{\pi t})$, then we have $\widetilde{f}_1(\varphi) = f_1(t)$. And as φ travels from $-\pi$ to 0, then t goes from $+\infty$ to $-\infty$ on the real line. Similarly, we have

$$\sin\varphi = \frac{-1}{\cosh\pi t},\ \mathrm{d}\varphi = \frac{-\pi}{\cosh\pi t}\mathrm{d}t.$$

Therefore, we can obtain

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta-\varphi)\widetilde{f}_1(\varphi)\mathrm{d}\varphi = \frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\widetilde{f}_1(\varphi)\mathrm{d}\varphi = \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_1(t)}{\cosh\pi t+\cos\pi y}\mathrm{d}t.$$

Assignment 13.4 (Exercise 8.5.8)

Find a harmonic function u in the open first quadrant that extends continuously up to the boundary except at the points 0 and 1, and that takes on the following prescribed boundary values: $u(x,y)=1$ on the half-lines $\{y=0, x>1\}$ and $\{x=0, y>0\}$, and $u(x,y)=0$ on the segment $\{0<x<1, y=0\}$.

[Hint: Find the conformal maps $F_1, F_2, \cdots, F_5$ indicated in Figure 11. Note that $\arg(z)/\pi$ is harmonic on the upper half-plane, equals 0 on the positive real axis, and 1 on the negative real axis.]

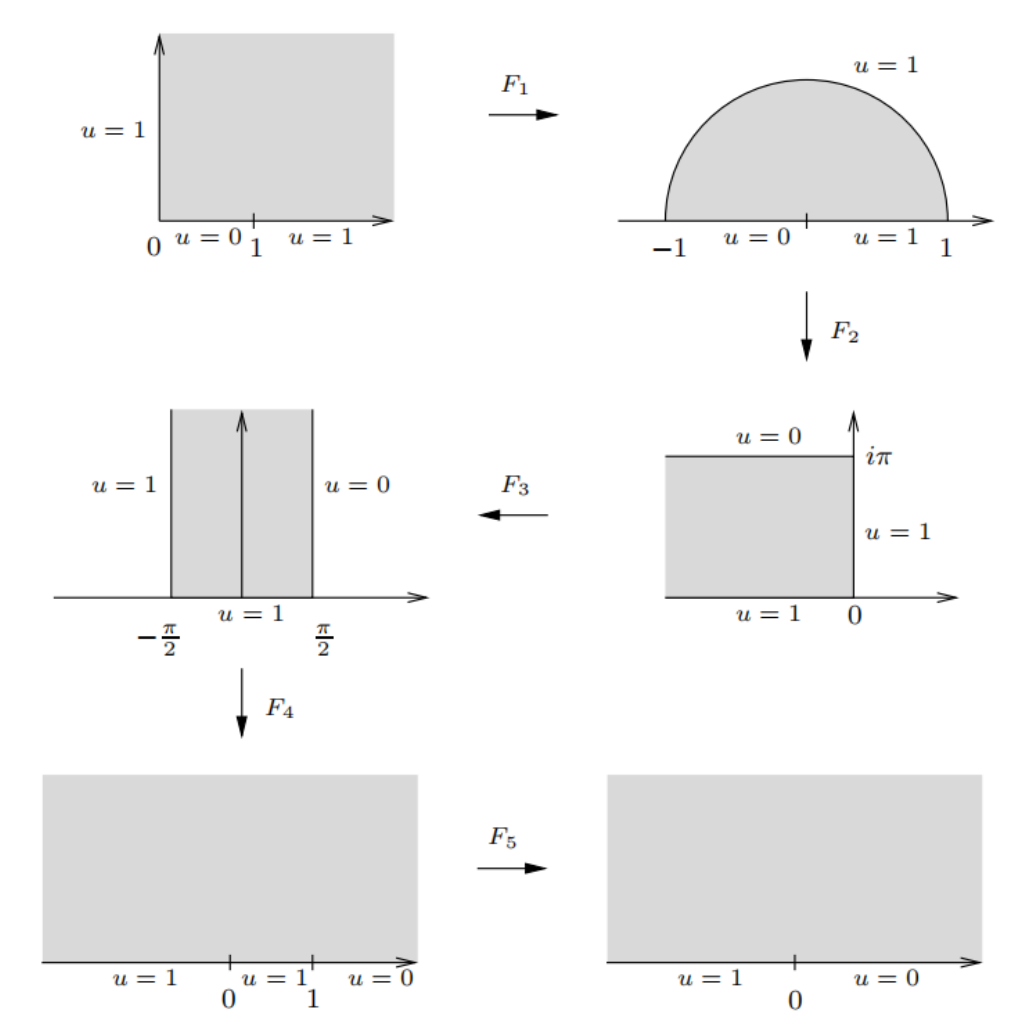

Figure 11. *Successive conformal maps in Exercise 8*

Solution *Let $F_1(z)=(z-1)/(z+1)$, $F_2(z)=\log z$, $F_3(z)=-iz-\pi/2$, $F_4(z)=\sin z$, $F_5(z)=z-1$, and let $\Phi=F_5\circ F_4\circ F_3\circ F_2\circ F_1$. Then Φ maps the open first quadrant conformally to the upper half-plane $\mathbb{H}$, and the required boundary pieces are mapped respectively to the positive and negative real axes. We consider the function $\widetilde{u}(z)=\arg(z)/\pi$ defined in $\mathbb{H}$, where $0<\arg(z)<\pi$ in $\mathbb{H}$. Hence $\widetilde{u}(z)$ takes the value 0 on the positive real axis and 1 on the negative real axis. Since $\widetilde{u}(z)=\operatorname{Im}(\log z/\pi)$, where $\log z$ is the principal branch restricted to $\mathbb{H}$, $\widetilde{u}(z)$ is harmonic in $\mathbb{H}$. Define $u(z)=\widetilde{u}(\Phi(z))$, then $u(z)$ is harmonic in the open first quadrant by Lemma 1.3 of the textbook and it* *has the required boundary behavior. Thus, u is the desired function.*

Assignment 13.5 (Exercise 8.5.9)

Prove that the function u defined by

$$u(x, y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right) \text{ and } u(0,1) = 0$$

is harmonic in the unit disc and vanishes on its boundary. Note that u is not bounded in $\mathbb{D}$. ♠

Proof The function $(i+z)/(i-z)$ is holomorphic in $\mathbb{D}$, since $i \notin \mathbb{D}$. Hence its real part is harmonic in $\mathbb{D}$. For $z = x + iy$, a direct computation gives

$$u(x, y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right) = \frac{1-|z|^2}{|i-z|^2}.$$

Therefore, u vanishes on $\partial\mathbb{D}$ except possibly at $z = i$. But the definition $u(0, 1) = 0$ gives the required boundary value at that point. Finally, u is not bounded in $\mathbb{D}$. Indeed, for $0 < r < 1$, we have

$$u(0, r) = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r} \to +\infty \text{ as } r \to 1^-.$$

Assignment 13.6 (Exercise 8.5.11)

Show that if $f : D(0, R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leqslant M$ for some $M > 0$, then

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{MR}.$$

[Hint: Use the Schwarz lemma.] [If $f(z) \equiv c$, the quotient is defined only when $|c| < M$.] ♠

Proof If f is constant, the result is immediate. Assume that f is not constant. Then by the maximum modulus principle, $|f(z)| < M$ for all $z \in D(0, R)$. Define $g : \mathbb{D} \to \mathbb{D}$ by $g(\zeta) = f(R\zeta)/M$. Let

$$\psi_\alpha : \mathbb{D} \to \mathbb{D},\ \psi_\alpha(w) = \frac{\alpha - w}{1 - \overline{\alpha}w}, \text{ where } \alpha \in \mathbb{D}.$$

Then $\psi_{g(0)} \circ g : \mathbb{D} \to \mathbb{D}$ is holomorphic and $\psi_{g(0)}(g(0)) = 0$. By the Schwarz lemma, we have

$$\left|\frac{g(\zeta) - g(0)}{1 - \overline{g(0)}g(\zeta)}\right| \leqslant |\zeta|.$$

Taking $\zeta = z/R$ and substituting $g(\zeta) = f(z)/M$, we can obtain

$$\left|\frac{M(f(z) - f(0))}{M^2 - \overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{R} \Rightarrow \left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{MR}.$$

Assignment 13.7 (Exercise 8.5.13)

*The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by*

$$\rho(z, w) = \left|\frac{z - w}{1 - \overline{w}z}\right|.$$

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \leqslant \rho(z, w) \text{ for all } z, w \in \mathbb{D}.$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic *distance*

$$\rho(f(z), f(w)) = \rho(z, w) \text{ for all } z, w \in \mathbb{D}.$$

[Hint: Consider the automorphism $\psi_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha} z)$ and apply the Schwarz lemma to $\psi_{f(w)} \circ f \circ \psi_w^{-1}$.]

(b) Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2} \text{ for all } z \in \mathbb{D}.$$

This result is called the Schwarz-Pick lemma. See Problem 3 for an important application of this lemma.

Proof (a) For $\alpha \in \mathbb{D}$, let $\psi_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha} z)$. Fix $w \in \mathbb{D}$ and consider $h = \psi_{f(w)} \circ f \circ \psi_w^{-1}$. Hence we have $h : \mathbb{D} \to \mathbb{D}$ is holomorphic and $h(0) = 0$. By the Schwarz lemma, $|h(\zeta)| \leqslant |\zeta|$ for all $\zeta \in \mathbb{D}$. Taking $\zeta = \psi_w(z)$ gives $|\psi_{f(w)}(f(z))| \leqslant |\psi_w(z)|$. This is exactly $\rho(f(z), f(w)) \leqslant \rho(z, w)$. It holds for all $z, w \in \mathbb{D}$. And if f is an automorphism of $\mathbb{D}$, applying the same inequality to f^{-1} gives the reverse inequality. Therefore $\rho(f(z), f(w)) = \rho(z, w)$ for all $z, w \in \mathbb{D}$.

(b) From (a), for all $z, w \in \mathbb{D}$ and $z \neq w$, we have

$$\left| \frac{f(z) - f(w)}{z - w} \right| \leqslant \left| \frac{1 - \overline{f(w)} f(z)}{1 - \overline{w} z} \right|.$$

Letting $w \to z$, we can obtain the desired inequality:

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2} \text{ for all } z \in \mathbb{D}.$$

Assignment 13.8 (Problem 8.6.3)

The Schwarz-Pick lemma (see Exercise 13) is the infinitesimal version of an important classical observation in complex analysis and geometry.

*For complex numbers $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the **hyperbolic length** of w at z by*

$$\|w\|_z = \frac{|w|}{1 - |z|^2},$$

*where $|w|$ and $|z|$ denote the usual absolute values. This length is sometimes referred to as the **Poincaré metric**, and as a Riemann metric it is written as*

$$\mathrm{d}s^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

The idea is to think of w as a vector lying in the tangent space at z. Observe that for a fixed w, its hyperbolic length grows to infinity as z approaches the boundary of the disc. We pass from the infinitesimal hyperbolic length of tangent vectors to the global hyperbolic distance between two points by integration.

*(a) Given two complex numbers z_1 and z_2 in the disc, we define the **hyperbolic distance** between them by*

$$d(z_1, z_2) = \inf_\gamma \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \mathrm{d}t,$$

where the infimum is taken over all smooth curves $\gamma : [0,1] \to \mathbb{D}$ joining z_1 and z_2. Use the Schwarz-Pick lemma to prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d(f(z_1), f(z_2)) \leqslant d(z_1, z_2) \text{ for any } z_1, z_2 \in \mathbb{D}.$$

In other words, holomorphic functions are distance-decreasing in the hyperbolic metric.

(b) Prove that automorphisms of the unit disc preserve the hyperbolic distance, namely

$$d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2), \text{ for any } z_1, z_2 \in \mathbb{D}$$

and any automorphism φ. Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance, then either φ or $\overline{\varphi}$ is an automorphism of $\mathbb{D}$.

(c) Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism φ such that $\varphi(z_1) = 0$ and $\varphi(z_2) = s$ for some s on the segment $[0,1)$ on the real line.

(d) Prove that the hyperbolic distance between 0 and $s \in [0,1)$ is

$$d(0, s) = \frac{1}{2} \log \frac{1+s}{1-s}.$$

(e) Find a formula for the hyperbolic distance between any two points in the unit disc. ♠

Proof (a) Let $\gamma : [0,1] \to \mathbb{D}$ be a smooth curve joining z_1 and z_2. The Schwarz-Pick lemma gives

$$\frac{|f'(\gamma(t))|}{1-|f(\gamma(t))|^2} \leqslant \frac{1}{1-|\gamma(t)|^2} \Rightarrow \int_0^1 \frac{|(f\circ\gamma)'(t)|}{1-|f(\gamma(t))|^2}\mathrm{d}t \triangleq L(f\circ\gamma) \leqslant \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\mathrm{d}t.$$

Since $f \circ \gamma : [0,1] \to \mathbb{D}$ is a smooth curve joining $f(z_1)$ and $f(z_2)$, taking the infimum over all such γ, we can obtain $d(f(z_1), f(z_2)) \leqslant \inf_\gamma L(f\circ\gamma) \leqslant d(z_1, z_2)$ for any $z_1, z_2 \in \mathbb{D}$.

(b) If $\varphi \in \operatorname{Aut}(\mathbb{D})$, applying (a) to both φ and φ^{-1} gives $d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2)$. Conversely, we suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance. Then let $C_r(z_0) = \{z \in \mathbb{D} : d(z, z_0) = r\}$ be the hyperbolic circle centered at $z_0 \in \mathbb{D}$ with radius $r > 0$. Since $\psi_{z_0} \in \operatorname{Aut}(\mathbb{D})$, it preserves the hyperbolic distance and hence induces a bijection from $C_r(z_0)$ to $C_r(0)$, which is also a Euclidean circle since $d(0, w) > d(0, z)$ when $|w| > |z|$ and $d(0, w) = d(0, z)$ when $|w| = |z|$. And since $\psi_{z_0}^{-1}$ is a Möbius transformation, $C_r(z_0)$ is also a Euclidean circle if $C_r(z_0) \neq \varnothing$. By composing φ with $\psi_{\varphi(0)}$ and a rotation, we can assume $\varphi(0) = 0$ and $\varphi(1/2) = 1/2$. Now consider the hyperbolic circles centered at 0 and $1/2$ passing through $i/2$, they intersect exactly at $\pm i/2$. Therefore, $\varphi(i/2) = i/2$ or $\varphi(i/2) = -i/2$. By composing φ with conjugation if necessary, we can assume $\varphi(i/2) = i/2$. Since 0, $1/2$, $i/2$ are not collinear, we have $\varphi(z) = z$. Undoing these compositions, we conclude that either φ or $\overline{\varphi}$ is an automorphism of $\mathbb{D}$. So we are done.

(c) We can compose ψ_{z_1} with an appropriate rotation to get the desired automorphism φ.

(d) Let $\gamma(t) = x(t) + iy(t) : [0,1] \to \mathbb{D}$ be a smooth curve in $\mathbb{D}$ joining 0 to $s \in [0,1)$. Then

$$\int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\mathrm{d}t \geqslant \int_0^1 \frac{|x'(t)|}{1-|x(t)|^2}\mathrm{d}t = \int_0^1 \frac{|\mathrm{d}x(t)|}{1-|x(t)|^2} \geqslant \int_0^s \frac{\mathrm{d}u}{1-u^2} = \frac{1}{2}\log\frac{1+s}{1-s}.$$

The equality holds if we take $\gamma(t) = st$. This proves the desired result.

(e) By (b), (c), (d), and $|\psi_{z_1}(z_2)| = \rho(z_1, z_2) = |z_1 - z_2|/|1 - \overline{z_1}z_2|$, we have

$$d(z_1, z_2) = d(0, |\psi_{z_1}(z_2)|) = \frac{1}{2}\log\frac{1+\rho(z_1,z_2)}{1-\rho(z_1,z_2)} = \frac{1}{2}\log\frac{|1-\overline{z_1}z_2| + |z_1 - z_2|}{|1-\overline{z_1}z_2| - |z_1-z_2|}.$$

Chapter 14 Homework 14

Assignment 14.1 (Exercise 8.5.17)

If $\psi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha}z)$ for $|\alpha| < 1$, prove that

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha|^2 \mathrm{d}x\mathrm{d}y = 1 \text{ and } \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| \mathrm{d}x\mathrm{d}y = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2},$$

where in the case $\alpha = 0$ the expression on the right is understood as the limit as $|\alpha| \to 0$. [Hint: The first integral can be evaluated without a calculation. For the second, use polar coordinates, and for each fixed r use contour integration to evaluate the integral in θ.]

Proof Since $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$, by the change of variables formula we have

$$\pi = \mathrm{Area}(\psi_\alpha(\mathbb{D})) = \iint_{\psi_\alpha(\mathbb{D})} \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{D}} |\psi'_\alpha|^2 \mathrm{d}x\mathrm{d}y.$$

This proves the first identity. For the second identity, we have $|\psi'_\alpha(z)| = (1-|\alpha|^2)/|1-\overline{\alpha}z|^2$. By the rotational symmetry of the disc, we have

$$\iint_{\mathbb{D}} |\psi'_\alpha| \mathrm{d}x\mathrm{d}y = (1-|\alpha|^2)\iint_{\mathbb{D}} \frac{\mathrm{d}x\mathrm{d}y}{|1-|\alpha|z|^2}.$$

Using polar coordinates, we get

$$\iint_{\mathbb{D}} \frac{\mathrm{d}x\mathrm{d}y}{|1-|\alpha|z|^2} = \int_0^1\int_0^{2\pi} \frac{\rho\,\mathrm{d}\theta\mathrm{d}\rho}{1-2|\alpha|\rho\cos\theta+|\alpha|^2\rho^2}.$$

For fixed ρ, the inner integral is the integral of the Poisson kernel divided by $1-|\alpha|^2\rho^2$, hence

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2|\alpha|\rho\cos\theta+|\alpha|^2\rho^2} = \frac{2\pi}{1-|\alpha|^2\rho^2}.$$

Therefore

$$\iint_{\mathbb{D}} |\psi'_\alpha| \mathrm{d}x\mathrm{d}y = 2\pi(1-|\alpha|^2)\int_0^1 \frac{\rho\,\mathrm{d}\rho}{1-|\alpha|^2\rho^2} = \frac{\pi(1-|\alpha|^2)}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$$

After dividing by π, we obtain the desired formula. When $\alpha = 0$, the right-hand side is understood as the limit as $|\alpha| \to 0$, and this limit is 1.

Assignment 14.2 (Exercise 8.5.18)

Suppose that Ω is a simply connected domain that is bounded by a piecewise-smooth simple closed curve γ (in the terminology of Chapter 1). Then any conformal map F of $\mathbb{D}$ to Ω extends to a continuous bijection of $\overline{\mathbb{D}}$ to $\overline{\Omega}$. The proof is simply a generalization of the argument used in Theorem 4.2.

Proof Since γ is piecewise smooth, Ω has finite area and is locally path-connected at every point of its boundary. Therefore, the proofs of Lemmas 4.3-4.5 apply without change: the area estimate remains valid, and the curves used in the proof of Lemma 4.4 can still be chosen inside Ω near each boundary point. It follows that F extends continuously to a map $\overline{F} : \overline{\mathbb{D}} \to \overline{\Omega}$. Now we let $G = F^{-1} : \Omega \to \mathbb{D}$. The only geometric property of $\mathbb{D}$ used in the preceding argument is that, for every point $z_0 \in \partial\mathbb{D}$, the intersection of $\mathbb{D}$ with any sufficiently small circle centered at z_0 is an arc. Since $\partial\Omega = \gamma$ is piecewise smooth and simple, the same property holds at every point of $\partial\Omega$, including the points at which two smooth pieces meet. Thus the same argument applies to G, and G extends continuously to a map $\overline{G}:\overline{\Omega}\to\overline{\mathbb{D}}$. If $z\in\partial\mathbb{D}$, choose a sequence $\{z_n\}\subset\mathbb{D}$ with $z_n\to z$. Since $G(F(z_n))=z_n$, continuity gives $\overline{G}(\overline{F}(z))=\lim_{n\to\infty}G(F(z_n))=\lim_{n\to\infty}z_n=z$. Similarly, $\overline{F}(\overline{G}(w))=w$ for every $w\in\overline{\Omega}$. Hence $\overline{F}$ and $\overline{G}$ are inverses of each other, so $\overline{F}$ is a continuous bijection from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.

Assignment 14.3 (Exercise 8.5.20)

Other examples of elliptic integrals providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$\int_0^z \frac{\mathrm{d}\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}},\ \textit{with } \lambda\in\mathbb{R} \textit{ and } \lambda\neq 0,1$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda=-1$, the image of

$$\int_0^z \frac{\mathrm{d}\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are $\Gamma^2(1/4)/(2\sqrt{2\pi})$.

Proof (a) This follows from the Schwarz-Christoffel formula. The singularities of the integrand are at 0, 1, λ, and ∞. At each of these four points the corresponding interior angle is $\pi/2$. Therefore the image of the upper half-plane is a rectangle whose vertices are the images of 0, 1, λ, and ∞.

(b) Let $\lambda=-1$. The two adjacent side lengths of the rectangle are

$$\int_0^1 \frac{\mathrm{d}\zeta}{\sqrt{\zeta(1-\zeta^2)}} \text{ and } \int_1^\infty \frac{\mathrm{d}\zeta}{\sqrt{\zeta(\zeta^2-1)}}.$$

For the second integral, using the substitution $\zeta=1/x$, we obtain

$$\int_1^\infty \frac{\mathrm{d}\zeta}{\sqrt{\zeta(\zeta^2-1)}}=\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x^2)}}.$$

Thus the rectangle is a square. Its side length is

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x^2)}}.$$

Using $x=\sin\theta$, we get

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x^2)}}=\int_0^{\pi/2}(\sin\theta)^{-1/2}\mathrm{d}\theta=\frac{1}{2}B(1/4,1/2)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)}.$$

Since $\Gamma(1/4)\Gamma(3/4)=\sqrt{2}\pi$, this side length is $\Gamma^2(1/4)/(2\sqrt{2\pi})$.

Assignment 14.4 (Exercise 8.5.21)

We consider conformal mappings to triangles.

(a) Show that

$$\int_0^z \zeta^{-\beta_1}(1-\zeta)^{-\beta_2}\mathrm{d}\zeta,$$

with $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of 0, 1, and ∞, and with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is

$$\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Proof (a) This is a direct application of the Schwarz-Christoffel formula. The derivative of the mapping is $z^{-\beta_1}(1-z)^{-\beta_2}$. Thus the prevertices 0, 1, and ∞ correspond to the vertices of a triangle. The angles at the images of 0 and 1 are $(1-\beta_1)\pi$ and $(1-\beta_2)\pi$. If we set $\alpha_j+\beta_j = 1$ and $\beta_1+\beta_2+\beta_3 = 2$, then the third angle is $\alpha_3\pi$. Therefore the image is a triangle with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$.

(b) If $\beta_1 + \beta_2 = 1$, then the image rays of $(-\infty, 0]$ and $[1, \infty)$ are parallel. Thus the limiting triangle degenerates into a semi-infinite strip.

(c) If $0 < \beta_1 + \beta_2 < 1$, then the image rays of $(-\infty, 0]$ and $[1, \infty)$ meet only after being extended in the opposite directions. Thus the image is an unbounded polygonal region rather than an ordinary triangle.

(d) We compute the side lengths. The side opposite the angle $\alpha_3\pi$ has length

$$\int_0^1 z^{-\beta_1}(1-z)^{-\beta_2}\mathrm{d}z = B(1-\beta_1, 1-\beta_2) = B(\alpha_1, \alpha_2).$$

Hence we have

$$B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(1-\alpha_3)} = \frac{\sin(\alpha_3\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Similarly, using $z = 1/x$ for the side adjacent to $[1, \infty)$, we get

$$\int_1^\infty z^{-\beta_1}(z-1)^{-\beta_2}\mathrm{d}z = B(\alpha_2, \alpha_3) = \frac{\sin(\alpha_1\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Using $z = -x/(1-x)$ for the side adjacent to $(-\infty, 0]$, we get

$$\int_{-\infty}^0 (-z)^{-\beta_1}(1-z)^{-\beta_2}\mathrm{d}z = B(\alpha_3, \alpha_1) = \frac{\sin(\alpha_2\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Thus the side opposite the angle $\alpha_j\pi$ has length

$$\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Assignment 14.5 (Exercise 9.3.1)

Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1 \in \mathbb{R}$.

(a) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0 = (1/q)\omega_1$.

[Hint: Since p and q are relatively prime, there exist integers m and n such that $mq + np = 1$ (Corollary 1.3, Chapter 8, Book I).]

(b) *If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m - n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m, n range over the integers.*

Proof (a) Suppose $\omega_2/\omega_1 = p/q$, where p and q are relatively prime. Let $\omega_0 = \omega_1/q$. Then $q\omega_0 = \omega_1$ and $p\omega_0 = \omega_2$, so if f has period ω_0, then it has periods ω_1 and ω_2. Conversely, by Bezout's identity, there exist integers m and n such that $mq+np = 1$. Therefore $m\omega_1+n\omega_2 = (mq+np)\omega_0 = \omega_0$. Since integer linear combinations of periods are again periods, ω_0 is a period of f. Thus the two periodicity assumptions are equivalent.

(b) Suppose ω_2/ω_1 is irrational. By Kronecker's approximation theorem, the point set $\{m - n\omega_2/\omega_1 : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Thus the set of periods $\{m\omega_1 - n\omega_2 : m, n \in \mathbb{Z}\}$ is dense on the real line spanned by ω_1. In particular, there are non-zero periods tending to 0. A non-constant meromorphic function cannot have non-zero periods tending to 0. Indeed, if $T_j \to 0$ are periods, then at every point z where f is holomorphic we have $f(z + T_j) = f(z)$ for all large j. Thus

$$f'(z) = \lim_{j\to\infty} \frac{f(z + T_j) - f(z)}{T_j} = 0.$$

Also, f cannot have a pole, since translating one pole by the periods T_j would produce infinitely many poles accumulating at that pole. Hence f is entire and satisfies $f' = 0$, so f is constant.

Assignment 14.6 (Exercise 9.3.3)

In contrast with the result in Lemma 1.5, prove that the series

$$\sum_{n+m\tau\in\Lambda^*} \frac{1}{|n + m\tau|^2}, \text{ where } \tau \in \mathbb{H}$$

does not converge. In fact, show that

$$\sum_{1\leqslant n^2+m^2\leqslant R^2} \frac{1}{n^2 + m^2} = 2\pi \log R + O(1) \text{ as } R \to \infty.$$

Proof Since $\tau \in \mathbb{H}$, the two norms $|(n, m)| = (n^2 + m^2)^{1/2}$ and $|n + m\tau|$ are comparable on $\mathbb{R}^2$. So

$$\sum_{n+m\tau\in\Lambda^*} \frac{1}{|n + m\tau|^2} < \infty \iff \sum_{(n,m)\neq(0,0)} \frac{1}{n^2 + m^2} < \infty.$$

Since the RHS diverges, so does the LHS. So it remains to prove the stated asymptotic. Compare the sum with the integral of $(x^2 + y^2)^{-1}$ over the annulus $1 \leqslant x^2 + y^2 \leqslant R^2$. The usual unit-square comparison gives an error bounded independently of R, because the gradient of $(x^2+y^2)^{-1}$ is $O(r^{-3})$ on the circle of radius r, and summing over shells gives a convergent error. Therefore we have

$$\sum_{1\leqslant n^2+m^2\leqslant R^2} \frac{1}{n^2 + m^2} = \iint_{1\leqslant x^2+y^2\leqslant R^2} \frac{\mathrm{d}x\mathrm{d}y}{x^2 + y^2} + O(1) = 2\pi \log R + O(1) \text{ as } R \to \infty.$$

Assignment 14.7 (Exercise 9.3.5)

Let $\sigma(z)$ be the canonical product

$$\sigma(z) = z \prod_{j=1}^{\infty} E_2(z/\tau_j),$$

where τ_j is an enumeration of the periods $\{n+m\tau\}$ with $(n,m)\neq(0,0)$, and $E_2(z)=(1-z)\mathrm{e}^{z+z^2/2}$.

(a) Show that $\sigma(z)$ is an entire function of order 2 that has simple zeros at all the periods $n+m\tau$, and vanishes nowhere else.

(b) Show that

$$\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{(n,m)\neq(0,0)}\left[\frac{1}{z-n-m\tau}+\frac{1}{n+m\tau}+\frac{z}{(n+m\tau)^2}\right],$$

and that this series converges whenever z is not a lattice point.

(c) Let $L(z)=-\sigma'(z)/\sigma(z)$. Then

$$L'(z)=\frac{(\sigma'(z))^2-\sigma(z)\sigma''(z)}{(\sigma(z))^2}=\wp(z).$$

Proof (a) Let $K\subset\mathbb{C}$ be compact. For $z\in K$ and $|\tau_j|$ sufficiently large, Chapter 5, Lemma 4.2 gives $|E_2(z/\tau_j)-1|\leqslant C_K|z|^3/|\tau_j|^3$. Since $\sum_j|\tau_j|^{-3}$ converges, the product $\prod_{j=1}^{\infty}E_2(z/\tau_j)$ converges uniformly on compact subsets. Hence σ is entire. Its zeros are exactly the lattice points $n+m\tau$, and they are simple. It remains to determine the order of σ. Let $r=|z|\geqslant 2$ and fix $0<\varepsilon<1$. Since $|\tau_j|$ is comparable to the Euclidean norm of the corresponding lattice point, summing over annuli gives

$$\#\{j:|\tau_j|\leqslant 2r\}=O(r^2),\quad \sum_{|\tau_j|\leqslant 2r}\frac{1}{|\tau_j|}=O(r),\quad \sum_{|\tau_j|\leqslant 2r}\frac{1}{|\tau_j|^2}\asymp\log r.$$

We split the product according as $|\tau_j|\leqslant 2r$ or $|\tau_j|>2r$. If $|\tau_j|>2r$, then $|z/\tau_j|<1/2$, and the power series for $\log E_2$ gives $|E_2(w)|\leqslant\exp(C|w|^3)$ for $|w|\leqslant 1/2$. Hence

$$\prod_{|\tau_j|>2r}\left|E_2\left(\frac{z}{\tau_j}\right)\right|\leqslant\exp\left(Cr^3\sum_{|\tau_j|>2r}\frac{1}{|\tau_j|^3}\right)\leqslant\exp\left(C_\varepsilon r^{2+\varepsilon}\right)$$

where we used $\sum_j|\tau_j|^{-2-\varepsilon}<\infty$. For the remaining factors, we have

$$|E_2(w)|\leqslant(1+|w|)\exp\left(|w|+\frac{|w|^2}{2}\right).$$

Therefore, we can obtain

$$\prod_{|\tau_j|\leqslant 2r}\left|E_2\left(\frac{z}{\tau_j}\right)\right|\leqslant\exp\left(\sum_{|\tau_j|\leqslant 2r}\left[\log\left(1+\frac{r}{|\tau_j|}\right)+\frac{r}{|\tau_j|}+\frac{r^2}{2|\tau_j|^2}\right]\right)\leqslant\mathrm{e}^{Cr^2\log r}\leqslant\mathrm{e}^{C_\varepsilon r^{2+\varepsilon}}.$$

It follows that $|\sigma(z)|\leqslant A_\varepsilon\exp\left(B_\varepsilon|z|^{2+\varepsilon}\right)$. Since $\varepsilon>0$ is arbitrary, σ has order at most 2. On the other hand, if its order were $\rho_0<2$, we could choose $\rho_0<\rho<2$, and Theorem 2.1 with $s=2$ would imply $\sum_j|\tau_j|^{-2}<\infty$, a contradiction. Thus σ has order exactly 2.

(b) Taking logarithms, we have locally uniformly away from the lattice

$$\log\sigma(z)=\log z+\sum_{(n,m)\neq(0,0)}\left[\log\left(1-\frac{z}{n+m\tau}\right)+\frac{z}{n+m\tau}+\frac{z^2}{2(n+m\tau)^2}\right].$$

Thus we may differentiate term by term and obtain

$$\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{(n,m)\neq(0,0)}\left[\frac{1}{z-n-m\tau}+\frac{1}{n+m\tau}+\frac{z}{(n+m\tau)^2}\right].$$

The same local uniform convergence also proves that this series converges whenever z is not a lattice point, as desired.

(c) Let $L(z) = -\sigma'(z)/\sigma(z)$. Differentiating the formula in (b), we get

$$L'(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2}\right].$$

This is precisely the Weierstrass $\wp$-function. On the other hand, we have

$$L'(z) = \frac{(\sigma'(z))^2 - \sigma(z)\sigma''(z)}{(\sigma(z))^2}.$$

Therefore, the desired formula follows from the above discussion, and the proof is complete.

Assignment 14.8 (Exercise 9.3.8)

Let

$$E_4(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4}$$

be the Eisenstein series of order 4.

(a) Show that $E_4(\tau) \to \pi^4/45$ *as* $\operatorname{Im}(\tau) \to \infty$.

(b) More precisely,

$$\left|E_4(\tau) - \frac{\pi^4}{45}\right| \leqslant c\mathrm{e}^{-2\pi t} \text{ if } \tau = x + it \text{ and } t \geqslant 1.$$

(c) Deduce that

$$\left|E_4(\tau) - \tau^{-4}\frac{\pi^4}{45}\right| \leqslant ct^{-4}\mathrm{e}^{-2\pi/t} \text{ if } \tau = it \text{ and } 0 < t \leqslant 1.$$

Proof (a) By Theorem 2.5 of the textbook, we have

$$E_4(\tau) = 2\zeta(4) + \frac{2(2\pi)^4}{3!}\sum_{r=1}^{\infty} \sigma_3(r)\mathrm{e}^{2\pi i r\tau}.$$

Since $\zeta(4) = \pi^4/90$ and $\mathrm{e}^{2\pi i r\tau} \to 0$ as $\operatorname{Im}(\tau) \to \infty$, and for $\operatorname{Im}(\tau) \geqslant 1$, $|\sigma_3(r)\mathrm{e}^{2\pi i r\tau}| \leqslant r^4\mathrm{e}^{-2\pi r}$, the dominated convergence theorem gives

$$\sum_{r=1}^{\infty} \sigma_3(r)\mathrm{e}^{2\pi i r\tau} \to 0 \text{ as } \operatorname{Im}(\tau) \to \infty \Rightarrow \lim_{\operatorname{Im}(\tau)\to\infty} E_4(\tau) = \frac{\pi^4}{45}.$$

(b) Let $\tau = x + it$ and $t \geqslant 1$. From the formula above, we have $|E_4(\tau) - \pi^4/45| \leqslant C\sum_{r=1}^{\infty} \sigma_3(r)\mathrm{e}^{-2\pi r t}$. Since $\sigma_3(r) \leqslant Cr^4$, it follows that $\sum_{r=1}^{\infty} \sigma_3(r)\mathrm{e}^{-2\pi r t} \leqslant C'\sum_{r=1}^{\infty} r^4\mathrm{e}^{-2\pi r t}$. Also, we have

$$\sum_{r=1}^{\infty} r^4\mathrm{e}^{-2\pi r t} = A\left(\sum_{r=1}^{\infty} \mathrm{e}^{-2\pi r t}\right)^{(4)} = A\left(\frac{\mathrm{e}^{-2\pi t}}{1-\mathrm{e}^{-2\pi t}}\right)^{(4)} = O(\mathrm{e}^{-2\pi t})$$

for $t \geqslant 1$. Thus we can obtain

$$\left|E_4(\tau) - \frac{\pi^4}{45}\right| \leqslant c\mathrm{e}^{-2\pi t}.$$

(c) We use the transformation formula that $E_4(\tau) = \tau^{-4}E_4(-1/\tau)$. If $\tau = it$ and $0 < t \leqslant 1$, then $-1/\tau = i/t$. Therefore, by (b), we have

$$\left|E_4(\tau) - \tau^{-4}\frac{\pi^4}{45}\right| = |\tau|^{-4}\left|E_4(-1/\tau) - \frac{\pi^4}{45}\right| \leqslant ct^{-4}\mathrm{e}^{-2\pi/t}.$$