第二章 行列式

§1 行列式的定义

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$ 的一组解 $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

· $n=2$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = \dfrac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \triangleq \dfrac{\Delta_2(\vec{b}, \vec{\alpha}_2)}{\Delta_2(\vec{\alpha}_1, \vec{\alpha}_2)} \\ x_2 = \dfrac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \triangleq \dfrac{\Delta_2(\vec{\alpha}_1, \vec{b})}{\Delta_2(\vec{\alpha}_1, \vec{\alpha}_2)} \end{cases}$$

$\vec{\gamma}_1 = \vec{\alpha}_2$

$\vec{\gamma}_2 = -\vec{\alpha}_1$

$$\Delta_2(\vec{\alpha}_1, \vec{\alpha}_2) \triangleq \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

· $\Delta_2(\vec{\alpha}_1, \vec{\alpha}_2) = -\Delta_2(\vec{\alpha}_2, \vec{\alpha}_1)$ 　　· $\Delta_2(\lambda\vec{\alpha}_1, \vec{\alpha}_2) = \lambda\Delta_2(\vec{\alpha}_1, \vec{\alpha}_2)$

· $\Delta_2(\vec{\alpha}_1 + \vec{\alpha}_2, \vec{\alpha}_3) = \Delta_2(\vec{\alpha}_1, \vec{\alpha}_3) + \Delta_2(\vec{\alpha}_2, \vec{\alpha}_3)$.

· $n=3$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 & (1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 & (2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 & (3) \end{cases}$$

令 $\vec{\alpha}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$ $\vec{\alpha}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$ $\vec{\alpha}_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$ $\vec{\beta} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\vec{\alpha}_1' = \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$ $\vec{\alpha}_2' = \begin{pmatrix} a_{22} \\ a_{32} \end{pmatrix}$ $\vec{\alpha}_3' = \begin{pmatrix} a_{23} \\ a_{33} \end{pmatrix}$ $\vec{\beta}' = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix}$

$$\begin{cases} a_{22}x_2 + a_{23}x_3 = b_2 - a_{21}x_1 & (2') \\ a_{32}x_2 + a_{33}x_3 = b_3 - a_{31}x_1 & (3') \end{cases}$$

$$\Longrightarrow \begin{cases} x_2 = \dfrac{\Delta_2(\vec{\beta}' - x_1\vec{\alpha}_1', \vec{\alpha}_3')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} = \dfrac{\Delta_2(\vec{\beta}', \vec{\alpha}_3')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} - x_1\dfrac{\Delta_2(\vec{\alpha}_1', \vec{\alpha}_3')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} \\ x_3 = \dfrac{\Delta_2(\vec{\alpha}_2', \vec{\beta}' - x_1\vec{\alpha}_1')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} = \dfrac{\Delta_2(\vec{\alpha}_2', \vec{\beta}')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} + x_1\dfrac{\Delta_2(\vec{\alpha}_1', \vec{\alpha}_2')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} \end{cases}$$

代入 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ (1) 得

$$\left[a_{11} - a_{12}\frac{\Delta_2(\vec{\alpha}_1', \vec{\alpha}_3')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} + a_{13}\frac{\Delta_2(\vec{\alpha}_1', \vec{\alpha}_2')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')}\right]x_1 = b_1 - a_{12}\frac{\Delta_2(\vec{\beta}', \vec{\alpha}_3')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')} + a_{13}\frac{\Delta_2(\vec{\beta}', \vec{\alpha}_2')}{\Delta_2(\vec{\alpha}_2', \vec{\alpha}_3')}$$

$$\Longrightarrow \begin{cases} x_1 = \dfrac{b_1\Delta_2(\vec\alpha_2',\vec\alpha_3') - a_{12}\Delta_2(\vec\beta',\vec\alpha_3') + a_{13}\Delta_2(\vec\beta',\vec\alpha_2')}{a_{11}\Delta_2(\vec\alpha_2',\vec\alpha_3') - a_{12}\Delta_2(\vec\alpha_1',\vec\alpha_3') + a_{13}\Delta_2(\vec\alpha_1',\vec\alpha_2')} \triangleq \dfrac{\Delta_3(\beta,\vec\alpha_2,\vec\alpha_3)}{\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3)} \\ x_2 = \dfrac{\Delta_3(\vec\alpha_1,\beta,\vec\alpha_3)}{\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3)} \\ x_3 = \dfrac{\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\beta)}{\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3)} \end{cases}$$

令 $\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- $\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3) = -\Delta_3(\vec\alpha_2,\vec\alpha_1,\vec\alpha_3) = -\Delta_3(\vec\alpha_1,\vec\alpha_3,\vec\alpha_2)$
- $\Delta_3(\lambda\vec\alpha_1,\vec\alpha_2,\vec\alpha_3) = \lambda\Delta_3(\vec\alpha_1,\vec\alpha_2,\vec\alpha_3)$

  $\Delta_3(\vec\alpha+\vec\beta,\vec\alpha_3,\vec\alpha_4) = \Delta_3(\vec\alpha,\vec\alpha_2,\vec\alpha_3) + \Delta_3(\vec\beta,\vec\alpha_2,\vec\alpha_3)$

设可定义 $\Delta_{n-1}(\vec\alpha_1,\vec\alpha_2,\cdots,\vec\alpha_{n-1}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \cdots & & & \cdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}$，其中 $\vec\alpha_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{n-1,i} \end{pmatrix}$，

使得 Δ_{n-1} 满足：

- $\Delta_{n-1}(\vec\alpha_1,\cdots,\overset{i}{\vec\alpha_i},\cdots,\overset{j}{\vec\alpha_j},\cdots) = -\Delta_{n-1}(\cdots\overset{i}{\vec\alpha_j},\cdots,\overset{j}{\vec\alpha_i},\cdots)$ （反对称）
- $\Delta_{n-1}(\cdots\lambda\vec\alpha_i,\cdots) = \lambda\Delta_{n-1}(\cdots\vec\alpha_i,\cdots)$
- $\Delta_{n-1}(\cdots,\vec\beta+\vec\gamma,\cdots) = \Delta_{n-1}(\cdots,\vec\beta,\cdots) + \Delta_{n-1}(\cdots,\vec\gamma,\cdots)$

多线性

且当方程组

$$\begin{cases} a_{11}x_1 + \cdots + a_{1,n-1}x_{n-1} = b_1 \\ \cdots \\ a_{n-1,1}x_1 + \cdots + a_{n-1,n-1}x_{n-1} = b_{n-1} \end{cases}$$

满足 $\Delta = \Delta_{n-1}(\vec\alpha_1,\cdots,\vec\alpha_{n-1}) \neq 0$ 时，方程组有解

$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{D_1}{\Delta} \\ \vdots \\ \frac{D_{n-1}}{\Delta} \end{pmatrix}$ 其中 $D_i = \Delta_{n-1}(\vec\alpha_1,\cdots\vec\alpha_{i-1},\vec\beta,\vec\alpha_{i+1},\cdots\vec\alpha_{n-1})$，$\vec\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$

现考察一般 n

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 & (1) \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 & (2) \\ \cdots & \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n & (n) \end{cases}$$

$$\vec{\alpha}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \cdots \vec{\alpha}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \quad \vec{\beta} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\vec{\alpha}_1' = \begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \cdots \vec{\alpha}_n' = \begin{pmatrix} a_{2,n} \\ \vdots \\ a_{n,n} \end{pmatrix} \quad \vec{\beta}' = \begin{pmatrix} b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} a_{22}x_2 + \cdots + a_{2n}x_n = b_2 - a_{21}x_1 & (2') \\ \vdots & \vdots \\ a_{n2}x_2 + \cdots + a_{nn}x_n = b_n - a_{n1}x_1 & (n') \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \dfrac{\Delta_{n-1}(\vec{\beta}' - x_1\vec{\alpha}_1', \vec{\alpha}_3', \cdots, \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} = \dfrac{\Delta_{n-1}(\vec{\beta}', \vec{\alpha}_3', \cdots, \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} - x_1 \dfrac{\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_3', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \vec{\alpha}_3', \cdots \vec{\alpha}_n')} \\ x_3 = \dfrac{\Delta_{n-1}(\vec{\alpha}_2', \vec{\beta}' - x_1\vec{\alpha}_1', \vec{\alpha}_4', \cdots, \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} = \dfrac{\Delta_{n-1}(\vec{\alpha}_2', \vec{\beta}', \vec{\alpha}_4', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots, \vec{\alpha}_n')} + x_1 \dfrac{\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_2', \vec{\alpha}_4', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots, \vec{\alpha}_n')} \\ \vdots \\ x_n = \dfrac{\Delta_{n-1}(\vec{\alpha}_2', \vec{\alpha}_3', \cdots, \vec{\alpha}_{n-1}', \vec{\beta}')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots, \vec{\alpha}_n')} + (-1)^{n+1} x_1 \dfrac{\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_2', \cdots, \vec{\alpha}_{n-1}')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots, \vec{\alpha}_n')} \end{cases}$$

代入 (1) 得

$$\left[a_{11} - a_{12}\frac{\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_3', \cdots, \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \vec{\alpha}_3', \cdots, \vec{\alpha}_n')} + a_{13}\frac{\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_2', \vec{\alpha}_4', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} + \cdots + (-1)^{n+1}a_{1n}\frac{\Delta_{n-1}(\vec{\alpha}_1', \cdots, \vec{\alpha}_{n-1}')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')}\right] x_1$$

$$= \left[b_1 - a_{12}\frac{\Delta_{n-1}(\vec{\beta}', \vec{\alpha}_3', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} - a_{13}\frac{\Delta_{n-1}(\vec{\alpha}_2', \vec{\beta}', \vec{\alpha}_4', \cdots \vec{\alpha}_n')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')} - \cdots - a_{1n}\frac{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_{n-1}', \vec{\beta}')}{\Delta_{n-1}(\vec{\alpha}_2', \cdots \vec{\alpha}_n')}\right]$$

$$\Rightarrow x_1 = \frac{a_{11}\Delta_{n-1}(\vec{\alpha}_2', \vec{\alpha}_3', \cdots \vec{\alpha}_n') - a_{12}\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_3', \cdots, \vec{\alpha}_n') + a_{13}\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_2', \vec{\alpha}_4', \cdots \vec{\alpha}_n') + \cdots + (-1)^{n+1}a_{1n}\Delta_{n-1}(\vec{\alpha}_1', \cdots \vec{\alpha}_{n-1}')}{b_1\Delta_{n-1}(\vec{\alpha}_2', \cdots, \vec{\alpha}_n') - a_{12}\Delta_{n-1}(\vec{\beta}', \vec{\alpha}_3', \cdots, \vec{\alpha}_n') + a_{13}\Delta_{n-1}(\vec{\beta}', \vec{\alpha}_2', \vec{\alpha}_4', \cdots, \vec{\alpha}_n') \cdots + (-1)^{n+1}a_{1n}\Delta_{n-1}(\vec{\beta}', \vec{\alpha}_2', \cdots, \vec{\alpha}_{n-1}')}$$

令 $\Delta_n(\vec{\alpha}_1, \cdots, \vec{\alpha}_n) \triangleq a_{11}\Delta_{n-1}(\vec{\alpha}_2', \vec{\alpha}_3', \cdots, \vec{\alpha}_n') - a_{12}\Delta_{n-1}(\vec{\alpha}_1', \vec{\alpha}_3', \cdots, \vec{\alpha}_n') + \cdots + (-1)^{n+1}a_{1n}\Delta_{n-1}(\vec{\alpha}_1', \cdots \vec{\alpha}_{n-1}')$

则
$$\begin{cases} x_1 = \dfrac{\Delta_n(\vec{\beta}, \vec{\alpha}_2, \cdots, \vec{\alpha}_n)}{\Delta_n(\vec{\alpha}_1, \vec{\alpha}_2 \cdots \vec{\alpha}_n)} \\ x_2 = \dfrac{\Delta_n(\vec{\alpha}_1, \vec{\beta}, \vec{\alpha}_3, \cdots \vec{\alpha}_n)}{\Delta_n(\vec{\alpha}_1, \vec{\alpha}_2, \cdots \vec{\alpha}_n)} \\ \vdots \\ x_n = \dfrac{\Delta_n(\vec{\alpha}_1, \cdots, \vec{\alpha}_{n-1}, \vec{\beta})}{\Delta_n(\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n)} \end{cases}$$

Δ_n 满足:

- $\Delta_n(\vec{\alpha}_1, \cdots, \vec{\alpha}_n) = -\Delta_n(\vec{\alpha}_1, \cdots, \vec{\alpha}_{i+1}, \vec{\alpha}_i, \cdots \vec{\alpha}_n)$
- $\Delta_n(\lambda\vec{\alpha}_1, \cdots \vec{\alpha}_n) = \lambda\Delta_n(\vec{\alpha}_1, \cdots, \vec{\alpha}_n)$
- $\Delta_n(\vec{\alpha}_1 + \vec{\beta}, \vec{\alpha}_2, \cdots \vec{\alpha}_n) = \Delta_n(\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n) + \Delta_n(\vec{\beta}, \vec{\alpha}_2, \cdots \vec{\alpha}_n)$

定义1.1 排成n行n列的n^2个数 a_{ij} $(i,j=1,\cdots,n)$ 按以下方式得到唯一数，称为一个行列式：

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ a_{32} & \cdots & a_{3n} \\ & \cdots & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ & \cdots & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+(-1)^{i+1} a_{1i}\begin{vmatrix} a_{21} & \cdots & a_{2,i-1} & a_{2,i+1} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3,i-1} & a_{3,i+1} & \cdots & a_{3n} \\ & & \cdots & \cdots & & \\ a_{n1} & & a_{n,i-1} & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix} + (-1)^{n+1} a_{1n}\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} \end{vmatrix}$$

注 · 对方阵 $A=(a_{ij})_{n\times n}$ 也可称作方阵的行列式，记作 $\det A$

即 $\det: \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}$

· 行列式可看作列向量空间或行向量空间上的多元函数，即

$\det: \mathbb{F}^{n\times 1} \times \cdots \times \mathbb{F}^{n\times 1} \longrightarrow \mathbb{F}$ 或

$\det: \mathbb{F}^{1\times n} \times \cdots \times \mathbb{F}^{1\times n} \longrightarrow \mathbb{F}$

例
$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

子式与余子式 $\quad A=(a_{ij})_{n\times n}$

子矩阵：
$$A\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_s \end{pmatrix} = \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_s} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_s} \\ \cdots & & \cdots \\ a_{i_r,j_1} & \cdots & a_{i_r,j_s} \end{pmatrix} \quad \begin{matrix} i_1<i_2<\cdots<i_r \\ j_1<j_2<\cdots<j_s \end{matrix}$$

子式
$$\left|A\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}\right| \quad 或 \quad A\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$$

(i,j)余子式
$$M_{ij} \triangleq A\begin{pmatrix} 1,\cdots,i-1,i+1,\cdots,n \\ 1,\cdots,j-1,j+1,\cdots,n \end{pmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \cdots & & & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

则 $$\det A = \sum_{j=1}^{n} a_{1j}A_{1j} = \sum_{j=1}^{n}(-1)^{1+j} a_{1j}M_{1j}$$

§2 行列式的性质

定理2.1 $A=(a_{ij})_{n\times n}\in \mathbb{F}^{n\times n}$ 则

$$\det A=\sum_{i=1}^{n} a_{ki}A_{ki}=\sum_{i=1}^{n}(-1)^{k+i}a_{ki}M_{ki}\qquad \forall\ 1\le k\le n$$

PF 对 n 归纳.

- $n=2$ 显然成立.
- 设结论对 $n-1$ 阶行列式成立. 由定义 $\det A=\sum_{i=1}^{n}(-1)^{1+i}a_{1i}M_{1i}$

固定 $2\le k\le n$　令 D_{ij} 为删去 A 的第 $1,k$ 行 第 i,j 列所得子式.

显然 $D_{ij}=D_{ji}$.

将 M_{1i} 按第 $k-1$ 行展开（M_{1i} 的第 $k-1$ 行落在 A 的第 k 行），

$$M_{1i}=\sum_{j=1}^{i-1}(-1)^{k-1+j}a_{kj}D_{ij}+\sum_{j=i+1}^{n}(-1)^{k-1+j-1}a_{kj}D_{ij}$$

$$\det A=\sum_{i=1}^{n}(-1)^{i+1}a_{1i}\left(\sum_{j=1}^{i-1}(-1)^{k-1+j}a_{kj}D_{ij}+\sum_{j=i+1}^{n}(-1)^{k+j}a_{kj}D_{ij}\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{i-1}(-1)^{k+i+j}a_{1i}a_{kj}D_{ij}+\sum_{i=1}^{n}\sum_{j=i+1}^{n}(-1)^{k+i+j+1}a_{1i}a_{kj}D_{ij}$$

$$=\sum_{1\le i<j\le n}(-1)^{k+i+j+1}(a_{1i}a_{kj}-a_{1j}a_{ki})D_{ij}$$

同理 $\sum_{i=1}^{n}(-1)^{k+i}a_{ki}M_{ki}$

$$=\sum_{i=1}^{n}(-1)^{k+i}a_{ki}\left(\sum_{j=1}^{i-1}(-1)^{j+1}a_{1j}D_{ij}+\sum_{j=i+1}^{n}(-1)^{j}a_{1j}D_{ij}\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{i-1}(-1)^{k+i+j+1}a_{ki}a_{1j}D_{ij}+\sum_{i=1}^{n}\sum_{j=i+1}^{n}(-1)^{k+i+j}a_{ki}a_{1j}D_{ij}$$

$$=\sum_{1\le j<i\le n}(-1)^{k+i+j+1}(a_{1j}a_{ki}-a_{1i}a_{kj})D_{ij}\qquad \#$$

推论2.2 $\det A=\sum_{1\le i<j\le n}(-1)^{1+k+i+j}\left|A\begin{pmatrix}1 & k\\ i & j\end{pmatrix}\right|D_{ij}$, 其中 $D_{ij}=\left|A\begin{pmatrix}\hat{1}\cdots\hat{k}\cdots n\\ 1\cdots\hat{i}\cdots\hat{j}\cdots n\end{pmatrix}\right|$.

例

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots a_{2n-1} & \\ \vdots & \iddots & & \\ a_{n1} & & & \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}a_{1n}a_{2n-1}\cdots a_{n1}$$

定理2.3 行列式具有如下性质.

(1) 交换 A 的两行得到 B　则 $\det A=-\det B$

(2) 将 A 的某行乘以 λ 倍得到 B　则 $\det B=\lambda\det A$

(3)
$$\begin{vmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ b_1+c_1 & & b_n+c_n\\ \vdots & & \vdots\\ a_{n1} & \cdots & a_{nn}\end{vmatrix}=\begin{vmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ b_1 & \cdots & b_n\\ \vdots & & \vdots\\ a_{n1} & \cdots & a_{nn}\end{vmatrix}+\begin{vmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ c_1 & \cdots & c_n\\ \vdots & & \vdots\\ a_{n1} & \cdots & a_{nn}\end{vmatrix}$$

(4) A的两行成比例，则 $\det A=0$. 特别地 A的两行相等，则有 $\det A=0$

(5) 将A的某行乘以 λ 倍加到另行得到 B，则 $\det A=\det B$.

PF (1) 由推论2.3可得

(2)、(3) 按第 k 行展开

(4) 由(1)，$\det A=-\det A \Rightarrow \det A=0$

(5) 由(3)、(4)可得.

推论2.4 $\sum_{j=1}^{n} a_{kj} A_{ij} = \begin{cases} \det A & k=i, \\ 0 & k\neq i. \end{cases}$

注 (1) $\det A$ 可看作关于 a_{ij} 的 n 次齐次多项式.

(2) $A=\begin{pmatrix} \vec{\alpha}_1 \\ \vdots \\ \vec{\alpha}_n \end{pmatrix}$ 则

$\det A = \det(\vec{\alpha}_1, \cdots, \vec{\alpha}_n)$. $\det: \mathbb{F}^{1\times n}\times\cdots\times\mathbb{F}^{1\times n}\to\mathbb{F}$,

满足 (1) $\det(\vec{\alpha}_1,\cdots\vec{\alpha}_i,\cdots\vec{\alpha}_j,\cdots\vec{\alpha}_n)=-\det(\vec{\alpha}_1,\cdots,\vec{\alpha}_j,\cdots\vec{\alpha}_i,\cdots\vec{\alpha}_n)$

(2) $\det(\vec{\alpha}_1,\cdots,\vec{\beta}+\vec{\gamma},\cdots\vec{\alpha}_n)=\det(\vec{\alpha}_1,\cdots,\vec{\beta},\cdots\vec{\alpha}_n)+\det(\vec{\alpha}_1,\cdots,\vec{\gamma},\cdots\vec{\alpha}_n)$

$\det(\vec{\alpha}_1,\cdots\lambda\vec{\alpha}_k,\cdots,\vec{\alpha}_n)=\lambda\det(\vec{\alpha}_1,\cdots\vec{\alpha}_k,\cdots\vec{\alpha}_n)$

(3) $\det(\vec{e}_1,\cdots,\vec{e}_n)=1$ $\quad \vec{e}_k=(0,\cdots 0,\underset{\substack{\uparrow\\ k}}{1},0\cdots 0)$

例
$$\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & & a \\ \cdots & & & \ddots & \\ a & a & a & \cdots & x \end{vmatrix}_{n\times n} = (x+(n-1)a)\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ & & \cdots & & \\ a & a & a & \cdots & x \end{vmatrix} = (x+(n-1)a)\cdot(x-a)^{n-1}$$

例
$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \\ 3 & 4 & 5 & \cdots & 2 \\ \cdots & & & & \\ n & 1 & 2 & \cdots & n-1 \end{vmatrix} = \frac{(n+1)n}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 3 & 4 & \cdots & 1 \\ 3 & 4 & 5 & \cdots & 2 \\ & & \cdots & & \\ n & 1 & 2 & \cdots & n-1 \end{vmatrix} = \frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 3 & 4 & \cdots n & 1 \\ 1 & 1 & 1 & 1 & 1-n \; 1 \\ \cdots & & & & \\ 1 & 1-n & 1 & 1 \cdots & 1 \end{vmatrix}$$

$$= \frac{n(n+1)}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n-1 \; 0 \\ & & & & -n \\ & & & \iddots & \\ & -n & & & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}}\frac{n(n+1)}{2}n^{n-2}$$

§3 行列式的展开

定理3.1（行列式完全展开）设 $A=(a_{ij})_{n\times n}\in\mathbb{F}^{n\times n}$ 则

$$\det A=\sum_{(j_1,\cdots,j_n)\in S_n}(-1)^{\tau(j_1,\cdots,j_n)}a_{1j_1}\cdots a_{nj_n}$$

PF
$$\det A=\det\left(\sum_{j_1=1}^{n}a_{1j_1}\vec{e}_{j_1},\ \cdots\cdots,\ \sum_{j_n=1}^{n}a_{nj_n}\vec{e}_{j_n}\right)$$
$$=\sum_{j_1,j_2,\cdots,j_n=1}^{n}a_{1j_1}a_{2j_2}\cdots a_{nj_n}\det(\vec{e}_{j_1},\cdots,\vec{e}_{j_n})$$
$$=\sum_{(j_1,\cdots,j_n)\in S_n}(-1)^{\tau(j_1,\cdots,j_n)}a_{1j_1}\cdots a_{nj_n}$$

S_n 为 $\{1,2,\cdots,n\}$ 的排列的全体.

$\tau(j_1,\cdots,j_n)=\#\{(s,t)\mid 1\le s<t\le n,\ j_s>j_t\}$ 为 $(j_1,\cdots,j_n)$ 的逆序数.

定理3.2 $\det A^T=\det A$，其中 $A^T=\begin{pmatrix}a_{11}&a_{21}&\cdots&a_{n1}\\a_{12}&a_{22}&\cdots&a_{n2}\\\vdots&&&\vdots\\a_{1n}&a_{2n}&\cdots&a_{nn}\end{pmatrix}$ 为 A 的转置.

PF
$$\det A^T=\sum_{(j_1,\cdots,j_n)\in S_n}(-1)^{\tau(j_1,\cdots,j_n)}a_{j_11}\cdots a_{j_nn}$$
$$=\sum_{(j_1,\cdots,j_n)\in S_n}(-1)^{\tau(j_1,\cdots,j_n)}a_{1k_1}\cdots a_{nk_n}$$ 其中 $(k_1,\cdots,k_n)$ 为 $(j_1,\cdots,j_n)$ 逆排列
$$=\sum_{(k_1,\cdots,k_n)\in S_n}(-1)^{\tau(k_1,\cdots,k_n)}a_{1k_1}\cdots a_{nk_n}=\det A$$

定理3.3 $\det A=\sum_{i=1}^{n}a_{ik}A_{ik}=\sum_{i=1}^{n}(-1)^{i+k}a_{ik}M_{ik}$，$\forall\ 1\le k\le n$

PF 对 $\det A^T$ 按第 k 行展开即可.

定理3.4
(1) 交换行列式两列，行列式变号
(2) 行列式某列乘以倍数加到另一列，行列式不变.
(3) 行列式关于列有多重线性.
(4) 行列式两列成比例，则行列式为0

例（Van der Mont 行列式）$x_1,\cdots,x_n\in\mathbb{F}$

$$V(x_1,x_2,\cdots,x_n)=\begin{vmatrix}1&1&\cdots&1\\x_1&x_2&\cdots&x_n\\\vdots&&&\vdots\\x_1^{n-1}&x_2^{n-1}&\cdots&x_n^{n-1}\end{vmatrix}$$

解. $V(x_1,\cdots,x_n) \xlongequal[\substack{\vdots \\ (2)-x_1(1)}]{\substack{(n)-x_1(n-1) \\ (n-1)-x_1(n-2)}} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & x_2-x_1 & & x_n-x_1 \\ & \vdots & & \\ & (x_2-x_1)x_2^{n-2} & & (x_n-x_1)x_n^{n-2} \end{vmatrix} = \begin{vmatrix} x_2-x_1 & \cdots & x_n-x_1 \\ (x_2-x_1)x_2 & & (x_n-x_1)x_n \\ \vdots & & \\ (x_2-x_1)x_2^{n-2} & & (x_n-x_1)x_n^{n-2} \end{vmatrix}$

$$= (x_2-x_1)\cdots(x_n-x_1)\, V(x_2,\cdots,x_n)$$

$$\xlongequal{\text{归纳}} (x_2-x_1)\cdots(x_n-x_1)(x_3-x_2)\cdots(x_n-x_2)\cdots\quad(x_n-x_1)$$

$$= \prod_{1\le i<j\le n}(x_j-x_i)$$

$V(x_1,\cdots,x_n)\neq 0 \iff x_1,\cdots,x_n$ 互不相同.

Recall $j_1,\cdots,j_n\in\mathbb{N}$ 两两不同. 数组 $(j_1,\cdots,j_n)$ 的

逆序数 $\tau(j_1,\cdots,j_n) \triangleq \#\{(s,t)\mid 1\le s<t\le n,\ j_s>j_t\}$

$= \#\{(t,s)\mid 1\le s<t\le n,\ j_t<j_s\}$

断言: 若 $(j_1,\cdots,j_n)$ 为 $(1,\cdots,n)$ 的一个排列. 则对 $\forall\, 1\le r\le n$

$\Rightarrow \tau(j_1,\cdots,j_n) = \tau(j_1,\cdots,j_r) + \tau(j_{r+1},\cdots,j_n) + k_1-1+\cdots+k_r-r$

$= \tau(j_1,\cdots,j_r) + \tau(j_{r+1},\cdots,j_n) + j_1+\cdots+j_r - \frac{r(r+1)}{2}$

其中 $k_1<k_2<\cdots<k_r$ 为 $j_1,\cdots,j_r$ 的排列

$\{(s,t)\mid 1\le s<t\le n,\ j_s>j_t\} = \{(s,t)\mid 1\le s<t\le r,\ j_s>j_t\} \sqcup \{(s,t)\mid r+1\le s<t\le n,\ j_s>j_t\}$

$\sqcup \{(s,t)\mid 1\le s\le r,\ r+1\le t\le n,\ j_s>j_t\}$

而 $\{(s,t)\mid 1\le s\le r,\ r+1\le t\le n,\ j_s>j_t\}$

$= \bigsqcup_{i=1,\cdots,r}\{(s,t)\mid 1\le s\le r,\ r+1\le t\le n,\ j_s=k_i>j_t\}$ $\quad \# = k_1-1+k_2-2+\cdots+k_r-r$

定理3.5 (Laplace展开)

$$\det A = \sum_{1\le k_1<\cdots<k_r\le n}(-1)^{i_1+\cdots+i_r+k_1+\cdots+k_r} A\begin{pmatrix} i_1 & \cdots & i_r \\ k_1 & \cdots & k_r\end{pmatrix} A\begin{pmatrix} i_{r+1} & \cdots & i_n \\ k_{r+1} & \cdots & k_n\end{pmatrix}$$

其中 $1\le i_1<\cdots<i_r\le n$ 为任意给定的正整数

$i_{r+1}<\cdots<i_n$ 为 $\{1,\cdots,n\}\setminus\{i_1,\cdots,i_r\}$ 的排列

$k_{r+1}<\cdots<k_n$ 为 $\{1,\cdots,n\}\setminus\{k_1,\cdots,k_r\}$ 的排列.

证明　首先对 $i_1=1, i_2=2, \dots, i_r=r$ 的情形证明

$$\det A = \sum_{(j_1,\dots,j_n)\in S_n} (-1)^{\tau(j_1,\dots,j_n)} a_{1j_1}\cdots a_{nj_n}$$

$$= \sum_{1\le k_1<\dots<k_r\le} \ \sum_{\substack{(j_1,\dots,j_r)\in S(k_1,\dots,k_r)\\(j_{r+1},\dots,j_n)\in S(k_{r+1},\dots,n)}} (-1)^{\tau(j_1,\dots,j_r)+\tau(j_{r+1},\dots,j_n)+k_1-1+\dots+k_r-r} a_{1j_1}\cdots a_{rj_r}a_{r+1,j_{r+1}}\cdots a_{nj_n}$$

$$= \sum_{1\le k_1<\dots<k_r} (-1)^{k_1-1+k_2-2+\dots+k_r-r} A\begin{pmatrix}1&\cdots&r\\k_1&\cdots&k_r\end{pmatrix} A\begin{pmatrix}r+1&\cdots&n\\k_{r+1}&\cdots&k_n\end{pmatrix}$$

一般的 $i_1<i_2<\dots<i_r$，可通过 $i_1-1+\dots+i_r-r$ 次交换相邻行将其化为 $1,\dots,r$ 行。 #

例
$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1r} & & & \\ \vdots & & \vdots & & & \\ a_{r1} & \cdots & a_{rr} & & & \\ & & & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ & * & & \vdots & & \vdots \\ & & & a_{n,r+1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr}\end{vmatrix} \begin{vmatrix} a_{r+1,r+1} & \cdots & a_{r+1,n} \\ \vdots & & \vdots \\ a_{n,r+1} & \cdots & a_{nn}\end{vmatrix}$$

例　若 A 的任一 r 阶子式均为 0，则 $|A|=0$

§4 Cramer 法则

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

有解 $\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$

$$\Longleftrightarrow c_1 \overset{\vec{\alpha}_1}{\begin{pmatrix}a_{11}\\a_{21}\\ \vdots\\a_{n1}\end{pmatrix}} + c_2 \overset{\vec{\alpha}_2}{\begin{pmatrix}a_{12}\\a_{22}\\ \vdots\\a_{n2}\end{pmatrix}} + \dots + c_n \overset{\vec{\alpha}_n}{\begin{pmatrix}a_{1n}\\a_{2n}\\ \vdots\\a_{nn}\end{pmatrix}} = \overset{\vec{\beta}}{\begin{pmatrix}b_1\\ \vdots\\b_n\end{pmatrix}}$$

向量积：
$$\mathbb{F}^{n\times1}\times\mathbb{F}^{n\times1}\longrightarrow\mathbb{F}$$
$$\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix}\times\begin{pmatrix}b_1\\ \vdots\\b_n\end{pmatrix}\longmapsto\left(\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix},\begin{pmatrix}b_1\\ \vdots\\b_n\end{pmatrix}\right)\triangleq a_1b_1+a_2b_2+\dots+a_nb_n$$

满足：
- $(\vec{\alpha},\vec{\beta}) = (\vec{\beta},\vec{\alpha})$
- $(\vec{\alpha},\lambda\vec{\beta}) = \lambda(\vec{\alpha},\vec{\beta})$
- $(\vec{\alpha},\vec{\beta}+\vec{\gamma}) = (\vec{\alpha},\vec{\beta}) + (\vec{\alpha},\vec{\gamma})$

观察：若存在 $\vec{\gamma}_1, \vec{\gamma}_2, \cdots, \vec{\gamma}_n \in \mathbb{F}^{n\times 1}$ 使得

$$(\vec{\gamma}_1, \vec{\alpha}_1) \neq 0, \quad (\vec{\gamma}_1, \vec{\alpha}_2) = 0, \quad \cdots, \quad (\vec{\gamma}_1, \vec{\alpha}_n) = 0$$
$$(\vec{\gamma}_2, \vec{\alpha}_1) = 0, \quad (\vec{\gamma}_2, \vec{\alpha}_2) \neq 0, \quad \cdots \quad (\vec{\gamma}_2, \vec{\alpha}_n) = 0$$
$$\vdots \qquad\qquad \vdots$$
$$(\vec{\gamma}_n, \vec{\alpha}_1) = 0 \quad (\vec{\gamma}_n, \vec{\alpha}_2) = 0, \quad \cdots \quad (\vec{\gamma}_n, \vec{\alpha}_n) \neq 0.$$

则有 $c_1 = \dfrac{(\vec{\beta}, \vec{\gamma}_1)}{(\vec{\gamma}_1, \vec{\alpha}_1)}, \quad c_2 = \dfrac{(\vec{\beta}, \vec{\gamma}_2)}{(\vec{\gamma}_2, \vec{\gamma}_2)}, \quad \cdots, \quad c_n = \dfrac{(\vec{\beta}, \vec{\gamma}_n)}{(\vec{\gamma}_n, \vec{\gamma}_n)}.$

对 A^T 应用推论 2.4 可知，取 $\vec{\gamma}_i = \begin{pmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{pmatrix}$ 即可。此时

$\forall \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^{n\times 1}$ 知，$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \vec{\gamma}_i \right) = \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & a_2 & a_{2,i+1} & \cdots & a_{2n} \\ & \cdots & & & & \cdots & \\ a_{n1} & \cdots & a_{n,i-1} & a_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$

定理 4.1（Cramer 法则）设方程组 $(*)$ 的系数矩阵行列式 $\Delta \neq 0$。令 Δ_i 为将 Δ 中第 i 行替换为 $\vec{\beta} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ 所得行列式，则方程组有唯一解 $\begin{pmatrix} \frac{\Delta_1}{\Delta} \\ \vdots \\ \frac{\Delta_n}{\Delta} \end{pmatrix}$

证明　若方程组有解，则按上述分析，取 $\vec{\gamma}_i = \begin{pmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{pmatrix}$ 即得解具有形式 $\begin{pmatrix} \frac{\Delta_1}{\Delta} \\ \vdots \\ \frac{\Delta_n}{\Delta} \end{pmatrix}$　下证解的存在性。

方法一：将 $\begin{pmatrix} \frac{\Delta_1}{\Delta} \\ \vdots \\ \frac{\Delta_n}{\Delta} \end{pmatrix}$ 代入方程组直接验证即可：只须证明对 $\forall i$，

$$a_{i1}\Delta_1 + a_{i2}\Delta_2 + \cdots + a_{in}\Delta_n = b_i \Delta$$

注意到 $\Delta_j = \sum_{k=1}^{n} b_k A_{kj} \quad \forall j$　故有

$$\text{左边} = \sum_{j=1}^{n} a_{ij}\Delta_j = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij} b_k A_{kj} = \sum_{k=1}^{n} b_k \sum_{j=1}^{n} a_{ij} A_{kj}$$
$$= \sum_{k=1}^{n} b_k (\delta_{ik}\Delta) = b_i \Delta$$

方法二：方程组经过系列初等变换可化为标准形式（定理 1.15）。由于 $\Delta \neq 0$，且初等行变换不改变系数矩阵行列式非 0 性，知 $r = n$，方程组有唯一解。

§5. 行列式计算

例1
$$\begin{vmatrix} a_1 & b_2 & & b_n \\ c_2 & a_2 & & \\ & & \ddots & \\ c_n & & & a_n \end{vmatrix}$$

解 方法一 (按最后一行展开)

$$\Delta = (-1)^{n+1} c_n \begin{vmatrix} b_2 & \cdots & b_n \\ a_2 & & \\ & \ddots & \\ & a_{n-1} & 0 \end{vmatrix} + a_n \begin{vmatrix} a_1 & b_2 & \cdots & b_{n-1} \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_{n-1} & & & a_{n-1} \end{vmatrix}$$

$$= -a_2 \cdots a_{n-1} b_n c_n + a_n \begin{vmatrix} a_1 & b_2 & \cdots & b_{n-1} \\ c_2 & a_2 & & \\ & & \ddots & \\ c_{n-1} & & & a_{n-1} \end{vmatrix}$$

$$= \cdots = a_1 \cdots a_n - a_2 \cdots a_{n-1} b_n c_n - a_2 \cdots a_{n-2} b_{n-1} c_{n-1} a_n$$
$$- \cdots - b_1 c_1 a_2 \cdots a_n$$

方法二 若 $a_2, \cdots, a_n$ 不为0 则

$$\Delta = \begin{vmatrix} a_1 - \frac{b_2 c_2}{a_2} - \cdots - \frac{b_n c_n}{a_n} & b_2 & \cdots & b_n \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix}$$
(分别用第 $2 \cdots n$ 列消去第1列中 $c_2, \cdots c_n$)

$$= \left(a_1 - \frac{b_2 c_2}{a_2} - \cdots - \frac{b_n c_n}{a_n}\right) a_2 \cdots a_n = a_1 \cdots a_n - \sum_{i=2}^{n} a_1 \cdots a_{i-1} b_i c_i a_{i+1} \cdots a_n$$

若某个 $a_i = 0$ 可用微扰法，或将行列式看作关于 a_i 的多项式

例2
$$\begin{vmatrix} \lambda_1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & \lambda_2 + a_2 b_2 & \cdots & a_2 b_n \\ \cdots & & \cdots & \\ a_n b_1 & a_n b_2 & \cdots & \lambda_n + a_n b_n \end{vmatrix}$$

解
$$\Delta = \begin{vmatrix} 1 & b_1 & b_2 & & b_n \\ & \lambda_1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ & a_2 b_1 & \lambda_2 + a_2 b_2 & \cdots & a_2 b_n \\ & \cdots & & \cdots & \\ & a_n b_1 & a_n b_2 & \cdots & \lambda_n + a_n b_n \end{vmatrix} = \begin{vmatrix} 1 & b_1 & b_2 & \cdots & b_n \\ -a_1 & \lambda_1 & & & \\ -a_2 & & \lambda_2 & & \\ \vdots & & & \ddots & \\ -a_n & & & & \lambda_n \end{vmatrix}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_n + \sum_{i=1}^{n} \lambda_1 \cdots \lambda_{i-1} a_i b_i \lambda_{i+1} \cdots \lambda_n$$

例3 $\Delta_n = \begin{vmatrix} a & b & & & \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{vmatrix}$

解 (递推) $\Delta_n = a \begin{vmatrix} a & c & & & \\ b & a & c & & \\ & b & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & b & a \end{vmatrix}_{n-1} - b \begin{vmatrix} c & b & & & \\ & a & b & & \\ & c & a & b & \\ & & \ddots & \ddots & b \\ & & & c & a \end{vmatrix}$

$= a\Delta_{n-1} - bc\,\Delta_{n-2}$

可取 $\Delta_1 = a$, $\Delta_0 = 1$

令 α, β 为 $x^2 - ax + bc = 0$ 两根. 则

$\Delta_n = \alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n$.

例4 $\Delta_n(x,y;a_1,\cdots,a_n) = \begin{vmatrix} a_1 & x & \cdots & & x \\ y & a_2 & \cdots & & x \\ & & \ddots & & \\ y & y & \cdots & & x \\ y & y & \cdots & y & a_n \end{vmatrix}$

解 (拆边) $\Delta_n(x,y;a_1\cdots a_n) = \begin{vmatrix} a_1 & x & \cdots & x & x \\ y & a_2 & & x & x \\ y & \cdots & & a_{n-1} & x \\ y & y & \cdots & y & x \end{vmatrix} + \begin{vmatrix} a_1 & x & \cdots & x & \\ y & a_2 & & x & \\ y & \cdots & & a_{n-1} & \\ y & y & \cdots & y & a_n - x \end{vmatrix}$

依次每行减下行 $= \begin{vmatrix} a_1-y & x-a_2 & & & & \\ & a_2-y & x-a_3 & & & \\ & & \ddots & \ddots & & \\ & & & & x-a_{n-1} & \\ & & & & a_{n-1}-y & 0 \\ y & & y & y \cdots & y & x \end{vmatrix} + (a_n-x)\Delta_{n-1}(x,y;a_1\cdots a_{n-1})$

$= x(a_1-y)\cdots(a_{n-1}-y) + (a_n-x)\Delta_{n-1}(x,y;a_1\cdots a_{n-1})$

$= x(a_{n-1}-y)\cdots(a_1-y) + (a_n-x)\,x\,(a_{n-2}-y)\cdots(a_1-y)$
$+ \cdots + (a_n-x)\cdots(a_{i+1}-x)\,x\,(a_{i-1}-y)\cdots(a_1-y)$
$+ \cdots + (a_n-x)\cdots(a_3-x)\,x\,(a_1-y)$
$+ (a_n-x)\cdots(a_2-x)\,a_1$

同理 $\Delta_n(x,y;a_1\cdots a_n) = y(a_1-x)\cdots(a_{n-1}-x) + (a_n-y)\Delta_{n-1}(x,y,a_1\cdots a_{n-1})$

$\Rightarrow \Delta_n = \dfrac{x(a_1-y)\cdots(a_n-y) - y(a_1-x)\cdots(a_n-x)}{x-y}$

[若 $x=y$, 情况如何?]

例5 $\Delta_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_n \\ a_1 & x & a_2 & \cdots & a_n \\ a_1 & a_2 & x & \cdots & a_n \\ & & & \ddots & \\ a_1 & a_2 & \cdots & & x \end{vmatrix}$

解 Δ_n 为关于 x 的 n 次多项式

· 显然 $x=a_i$ $\Delta_n=0$ $\forall i$ 故 $x-a_i \mid \Delta_n$

· 将各列加到第1列 $\Delta_n = \begin{vmatrix} x+a_1+\cdots+a_n & a_1 & a_2 & \cdots & a_n \\ x+a_1+\cdots+a_n & x & a_2 & \cdots & a_n \\ \vdots & & & & \\ x+a_1+\cdots+a_n & a_2 & a_3 & \cdots & x \end{vmatrix}$

$\Rightarrow x+a_1+\cdots+a_n \mid \Delta_n$

· 若 a_i 两两不同 $a_1+\cdots+a_n \neq -a_i$ $\forall i$

则 $\Delta_n(x) = c(x+a_1+\cdots+a_n)(x-a_1)\cdots(x-a_n)$，比较次数

知 c 为待定常数，比较两边 x^n 前系数知

$$\Delta_n(x) = (x+a_1+\cdots+a_n)(x-a_1)\cdots(x-a_n)$$

例6 (Van der Monde 行列式)

$$\Delta_n(x_1,\cdots,x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

解: Δ_n 为关于 $x_1,\cdots,x_n$ 的 $\frac{1}{2}n(n-1)$ 次齐次多项式.

显然 $x_j=x_i$ 时, $\Delta_n=0$. 故 $x_j-x_i \mid \Delta_n$. $\forall j>i$

$\Rightarrow \prod_{i<j}(x_j-x_i) \mid \Delta_n$ 比较次数知

$\Delta_n = c\cdot\prod_{i<j}(x_j-x_i)$ 其中 c 为常数

比较 $x_n^{n-1}x_{n-1}^{n-2}\cdots x_1$ 前系数知 $c=1$. 故

$$\Delta_n = \prod_{i<j}(x_j-x_i)$$

例7 (缺行 Van der Monde 行列式)

$$\Delta_k = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_n \\ x_1^{k-1} & x_2^{k-1} & & x_n^{k-1} \\ x_1^{k+1} & x_2^{k+1} & \cdots & x_n^{k+1} \\ \cdots & & \cdots & \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix}$$

解：易知 Δ_k 为

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n & x \\ & \cdots & & \cdots & & \\ x_1^k & x_2^k & x_3^k & \cdots & x_n^k & x^k \\ & \cdots & & \cdots & & \\ x_1^n & x_2^n & & \cdots & x_n^n & x^n \end{vmatrix}$$

x^k 一项的代数余子式.

$$\Delta = \prod_{1\le i<j\le n}(x_j-x_i)\cdot(x-x_1)\cdots(x-x_n)$$

$$= \cdots + (-1)^{n+k}\Delta_k x^k + \cdots$$

而 Δ 中 x^k 的系数为 $(-1)^{n-k}\prod_{1\le i<j\le n}(x_j-x_i)\cdot\sum_{1\le j_1<\cdots<j_{n-k}\le n} x_{j_1}\cdots x_{j_{n-k}}$

$$\Rightarrow \Delta_k = \prod_{1\le i<j\le n}(x_j-x_i)\cdot\sum_{1\le j_1<\cdots<j_{n-k}\le n} x_{j_1}\cdots x_{j_{n-k}}$$

例8 求

$$\Delta = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \frac{1}{a_1+b_3} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \frac{1}{a_2+b_3} & \cdots & \frac{1}{a_2+b_n} \\ \cdots & & & \cdots & \\ \frac{1}{a_n+b_1} & \frac{1}{a_n+b_2} & \frac{1}{a_n+b_3} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix}$$

问1

$$\Delta \xlongequal[\ell_n-\ell_1]{\ell_2-\ell_1,\cdots} \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_n} \\ \frac{a_2-a_1}{(a_1+b_1)(a_2+b_1)} & \frac{a_2-a_1}{(a_1+b_2)(a_2+b_2)} & & \frac{a_2-a_1}{(a_1+b_n)(a_2+b_n)} \\ \vdots & \vdots & & \vdots \\ \frac{a_n-a_1}{(a_1+b_1)(a_n+b_1)} & \frac{a_n-a_1}{(a_1+b_2)(a_n+b_2)} & & \frac{a_n-a_1}{(a_1+b_n)(a_n+b_n)} \end{vmatrix}$$

$$= \frac{(a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)}{(a_1+b_1)(a_1+b_2)\cdots(a_1+b_n)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_n} \\ \cdots & & \cdots & \\ \frac{1}{a_n+b_1} & \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix}$$

$$= \frac{\prod_{i\ge 1}(a_i - a_1)}{\prod_{j=1}^{n}(a_1+b_j)} \begin{vmatrix} 1 & & & \\ \frac{1}{a_2+b_1} & \frac{b_2-b_1}{(a_2+b_1)(a_2+b_2)} & \cdots & \frac{b_n-b_1}{(a_2+b_1)(a_2+b_n)} \\ \vdots & & & \\ \frac{1}{a_n+b_1} & \frac{b_n-b_1}{(a_n+b_1)(a_n+b_2)} & & \frac{b_n-b_1}{(a_n+b_1)(a_n+b_n)} \end{vmatrix}$$

$$= \frac{\prod_{i\ge 1}(a_i-a_1)(b_i-b_1)}{(a_1+b_1)\prod_{j=2}^{n}(a_j+b_1)(a_1+b_j)} \begin{vmatrix} \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_n} \\ \cdots & & \cdots \\ \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix}$$

$$= \cdots = \frac{\prod_{j>i}(a_j-a_i)(b_j-b_i)}{\prod_{i,j=1}^{n}(a_i+b_j)}$$

解2 $\Delta = \dfrac{1}{\prod_{i,j=1}^{n}(a_i+b_j)} \underbrace{\begin{vmatrix} (a_1+b_2)\cdots(a_1+b_n) & (a_1+b_1)(a_1+b_3)\cdots(a_1+b_n) & \cdots & (a_1+b_1)\cdots(a_1+b_{n-1}) \\ (a_2+b_2)\cdots(a_2+b_n) & (a_2+b_1)(a_2+b_3)\cdots(a_2+b_n) & \cdots & (a_2+b_1)\cdots(a_2+b_{n-1}) \\ \cdots & \cdots & & \\ (a_n+b_2)\cdots(a_2+b_n) & (a_n+b_1)(a_n+b_3)\cdots(a_n+b_n) & \cdots & (a_n+b_1)\cdots(a_n+b_{n-1}) \end{vmatrix}}_{=D}$

则 右边行列式 D 为关于 a_i, b_j $i,j=1,\cdots n$ 的 $n(n-1)$ 次多项式.

显然 a_j-a_i, b_j-b_i 两组元素均为 D 的因子 比较次数知

$D = c\prod_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)$，其中 c 为常数.

$$\Longrightarrow \quad \Delta = c\cdot\frac{\prod_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)}{\prod_{i,j=1}^{n}(a_i+b_j)}$$

令 $a_i = \frac{1}{2}+ix$, $b_j = \frac{1}{2}-jx$

则 $$\Delta = \begin{vmatrix} \frac{1}{1} & \frac{1}{1-x} & \cdots & \frac{1}{1-(n-1)x} \\ \frac{1}{1+x} & \frac{1}{1} & \cdots & \frac{1}{1-(n-2)x} \\ \cdots & & \cdots & \\ \frac{1}{1+(n-1)x} & \frac{1}{1+(n-2)x} & \cdots & \frac{1}{1} \end{vmatrix} = c\cdot\frac{\prod_{1\le i<j\le n} -(j-i)^2x^2}{\prod_{i,j=1}^{n}(1+(i-j)x)}$$

两边令 $x\to\infty$，取极限知 $c=1$ 故有

$$\Delta = \prod_{1\le i<j\le n}(a_j-a_i)(b_j-b_i) \Big/ \prod_{i,j=1}^{n}(a_i+b_j)$$